D 11162 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2016

(CUCBCSS—UG)

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

Time : Three Hours Maximum : 120 Marks

Section A

Answer all the **twelve** *questions.*

Each question carries 1 mark.

1. Find the identity element in the binary structure $<\mathbb{Q}, *>$ if it exists when $a * b = ab/5$ for all $a, b \in \mathbb{Q}$.
2. Define subgroup of a group.
3. Express the additive inverse of 21 in the group $<\mathbb{Z}_{75}, +_{75}>$ as a positive integer in $<\mathbb{Z}_{75}>$.
4. Fill in the blanks : Order of the group of symmetries of a square is ———.
5. Define a division ring.
6. How many elements are there in the ring of matrices $M_2(\mathbb{Z}_2)$?
7. Fill in the blanks : Order of the subgroup $A_7 \leq S_7$ is ———.
8. Fill in the blanks : One non-zero solution of $x^2 = 0$ in Z_{50} is ———.
9. Define index of a subgroup H in a group G.
10. What is the characteristic of the ring of real numbers under usual addition and multiplication ?
11. Define a cyclic group. Give an example of a non-cyclic group.
12. Write any two units in the ring of Guassian integers $\{a + ib : a, b \in \mathbb{Z}\}$.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *out of fourteen questions.*

Each question carries 4 marks.

13. Determine whether the set of all real square matrices of order n is a group under matrix multiplication or not. Justify your claim.
14. Establish any necessary and sufficient conditions for a set H to be a subgroup of a group G.

Turn over

15. Determine the number of group homomorphisms from $\mathbb{Z}$ into $\mathbb{Z}$.

16. What is an octic group ? Is it an abelian group ? Justify your claim.

17. Define a ring and give an example of a finite ring which is not an integral domain.

18. Show that the identity and inverse in a group are unique.

19. Give an example of a finite group with the identity element e where the equation $x^2 = e$ has more than two solutions. Prove your claim.

20. Give two examples of non-trivial proper subgroups of $\mathbb{Z}$.

21. Show that every field is an integral domain but not conversely.

22. State Lagranges theorem and prove any result which can be established as a corollary to it.

23. If H is a subgroup of index two in a finite group G, show that $H \trianglelefteq G$.

24. Show that arbitrary intersection of subgroups is a subgroup.

25. Find all the units in the ring $\mathbb{Z}_{10}$.

26. Show that the characteristic of an integral domain is either 0 or a prime p.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*

Each question carries 7 marks.

27. If G is a finite group with identity element e show that for any a in G there exists a positive integer n such that $a^n = e$.

28. Show that every permutation σ of a finite set is a product of disjoint cycles.

29. Show that the subset S of $M_n(\mathbb{R})$ consisting of all invertible $n \times n$ matrices under matrix multiplication is a group.

30. Show that every permutation σ of a finite set is a product of disjoint cycles.

31. Define an automorphism of a group. Show that all automorphisms of a group G form a group under function composition.

32. If a is an integer relatively prime to n, then show that $a^{\phi(n)} - 1$ is divisible by n.

33. Solve : $x^2 = i$ in S_3 where i is the identity.

34. Show that every finite integral domain is a field.

35. Show that if H and K are two normal subgroups of a group G with $H \cap K = \{e\}$, then $hk = kh$ for all $h \in H$ and $k \in K$.

(6 × 7 = 42 marks)

Section D

Answer any **two** *out of three questions.*
Each question carries 13 *marks.*

36. Let G be cyclic group with generator a. If the order of G is infinite, then show that G is isomorphic to $<\mathbb{Z}, +>$. If G has finite order n, then show that G is isomorphic to $<\mathbb{Z}_n, +_n>$.

37. (a) State and prove fundamental theorem for group homomorphism.

(b) Show that if a finite group G contains a non-trivial subgroup of index 2 in G, then G is not simple.

38. (a) Show that 15 divides the number $n^3 - n$ for every integer n.

(b) Define an inner automorphism of a group G and give an example.

(2 × 13 = 26 marks)