International Journal of Pure and Applied Mathematics
Volume 108 No. 3 2016, 581-589
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v108i3.9

ijpam.eu

T_1 GRAPHS

Seena V[1 §], Raji Pilakkat[2]

[1,2]Department of Mathematics
University of Calicut
Calicut University
Malappuram (District), PIN 673 635, Kerala, INDIA

Abstract: A simple graph G is said to be T_1 if for any two distinct vertices u and v of G, one of the following conditions hold:

1. At least one of u and v is isolated
2. There exist two edges e_1 and e_2 of G such that e_1 is incident with u but not with v and e_2 is incident with v but not with u.

In this paper we discuss T_1 graphs and some examples of it. This paper also deals with the sufficient conditions for join of two graphs, middle graph of a graph and corona of two graphs to be T_1. It proved that line graph of any T_1 graph is T_1. Moreover, the relations between T_1 graphs with its incidence matrix and its adjacency matrix is discussed.

AMS Subject Classification: 05C99

Key Words: T_1 graph, incidence matrix, adjacency matrix, line graph, corona, middle graph

1. Introduction

All the graphs considered here are finite and simple. In this paper we denote the set of vertices of G by $V(G)$, the set of edges of G by $E(G)$, the maximum degree of G by $\Delta(G)$ and the minimum degree of G by $\delta(G)$.

The *degree* [5] of a vertex v in graph G, denoted by $\deg(v)$, is the number of edges incident with v. A *pendant vertex* [7] in a graph G is a vertex of degree one. A vertex v is *isolated* [5] if $\deg(v) = 0$.

Received: December 4, 2016
Published: July 20, 2016

§Correspondence author

By an *empty graph* [2] we mean a graph with no edges. A simple graph is said to be *complete* [1] if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n. A graph is *bipartite* [2] if its vertex set can be partitioned into two subsets, X and Y so that every edge has one end in X and other end in Y; such a partition (X, Y) is called a *bipartition* of the bipartite graph. A simple bipartite graph is *complete* if each vertex of X is adjacent to all vertices of Y. A complete bipartite graph with $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. Given two graphs, G and H, we say H is an *induced subgraph*[3] of G if $V(H) \subseteq V(G)$, and two vertices of H are adjacent if and only if they are adjacent in G. In this case if $V(H) = S$, we write $H = G[S]$ or $H = \langle S \rangle$. The *union* [9] of two graphs G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *line graph* [9] $L(G)$ of a graph G, is the graph whose vertex set is $E(G)$ and edge set is $\{ef : e, f \in E(G) \textit{ and } e, f \text{ have a vertex in common}\}$. The *join* [4] of two graphs G_1 and G_2 denoted by $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. The *corona* [4] of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where i^{th} vertex of G_1 is adjacent to every vertex in i^{th} copy of G_2. The *ring sum* [8] of two graphs G_1 and G_2, denoted by $G_1 \oplus G_2$, is the graph consisting of the vertex set $V(G_1) \cup V(G_2)$ and of edges that are either in G_1 or G_2, but not in both. The *middle graph* [6] of $G = (V(G), E(G))$ is the graph $M(G) = (V(G) \cup E(G), E'(G))$, where $uv \in E'$ if and only if either u is a vertex of G and v is an edge containing u, or u and v are edges having a vertex in common.

2. T_1 Graphs

In this paper we introduce the concept of T_1 graphs.

Definition 1. A graph G is said to be a T_1 graph if for any two distinct vertices u and v of G, one of the following conditions hold:

1. At least one of u and v is isolated
2. There exist two edges e_1 and e_2 of G such that e_1 is incident with u but not with v and e_2 is incident with v but not with u.

The terminology 'T_1 graph' is used for this new concept, because if G is a T_1 graph, then the topology generated by the collection of all two point sets consisting of the end vertices of edges of G and singleton sets consisting of isolated vertices of G is a T_1 topology on $V(G)$.

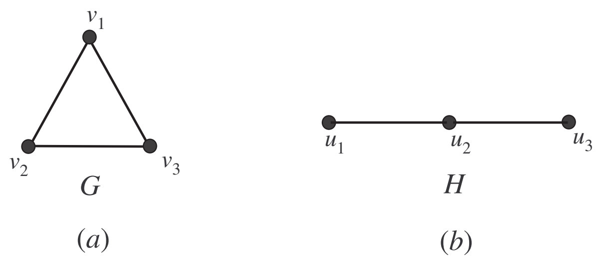

Figure 1: (a) An example of a T_1 graph. (b) An example of a non- T_1 graph.

Example 2.

The graph G in Figure 1, is T_1 where as the graph H in Figure 1, is not T_1. The failure of the graph H to be T_1 is that it contains a pendant edge.

Theorem 3. *Let G be a graph with $\delta(G) \geq 2$, then G is T_1.*

Proof. Let u and v be two distinct vertices of G. Since $\delta(G) \geq 2$, both u and v are adjacent to at least two vertices of G. Let w be a vertex adjacent to u in G distinct from v. Then $e = uw$ is an edge of G incident with u but not with v. Similarly we can prove that there exists an edge f incident with v but not with u. □

From the definition of T_1 graphs we have,

1. if G is a T_1 graph with no isolated vertices, then any supergraph of G is T_1.
2. n-regular graphs are T_1 if $n \neq 1$
3. for $n \geq 3$, the cycle C_n is T_1.
4. the complete graph K_n is T_1 if $n \neq 2$
5. the complete bipartite graph K_{mn} is T_1 if $m \geq 2$ and $n \geq 2$

Let u be a pendant vertex of a graph G with pendant edge uv. In this case there exist no edge containing u but not v in G. Hence G is not T_1. Therefore, we have the following proposition

Proposition 4. *If G is a graph with $\delta(G) = 1$, then G is not T_1.*

Proposition 5. *The union of T_1 graphs is T_1.*

Let G be a graph with no pendant edges. Then we can write the vertex set of G as $V(G) = K \cup H$, where K contains all the isolated vertices of G and H contains all non-isolated vertices of G. Then the subgraph of G induced by K is an empty graph which is T_1. The subgraph of G induced by H is also T_1 since it is a graph with minimum degree ≥ 2. Therefore, G being the union of two T_1 graphs is T_1. Hence we have the following proposition.

Proposition 6. *If G is a graph with no pendant edges, then G is T_1.*

Theorem 7. *Let G_1 and G_2 be two isomorphic graphs. If G_1 is T_1, then G_2 is also T_1.*

Proof. Given that G_1 and G_2 are isomorphic. Therefore, there exist bijections $f : V_1 \to V_2$ and $g : E_1 \to E_2$, such that $g(uv) = f(u)f(v)$ for every $uv \in E_1$. Let u and v be two distinct vertices of G_2. Since f is a bijection there exist two distinct vertices x and y of G_1 such that $f(x) = u$ and $f(y) = v$. Since G_1 is T_1, there exists an edge e_1 of G which is incident with x but not with y. Let $p \neq y$ be a vertex of G_1 such that $e = xp$. Then $g(e) = f(x)f(p) = uf(p)$. Since f is a bijection $g(e)$ is an edge of G_2 incident with u but not incident with v. Similarly we can prove there exists an edge of G_2 incident with v but not incident with u. Therefore, G_2 is T_1. Hence the theorem. □

3. Incidence Matrix and Adjacency Matrix

Theorem 8. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots v_n\}$, and edge set $E(G) = \{e_1, e_2, \ldots e_m\}$. Let $M = (m_{ij})$ be its incidence matrix. Then G is a T_1 graph if and only if there does not exist an index i such that $\sum_{j=1}^{n} m_{ij} = 1$.*

Proof. By Proposition 4, 6, a graph G is T_1 if and only if it has no pendant edges. That is, if and only if degree of each vertex of G is different from 1. That is, if and only if the sum of elements of each row of its incidence matrix $\neq 1$ □

By the definition of the adjacency matrix $A = (a_{ij})$ of a graph G with vertex set $V(G) = \{v_1, v_2, \ldots v_n\}$, $\sum_{j=1}^{n} a_{ij}$ will be the degree of the vertex v_i. We know that a graph G is T_1 if and only if $\delta(G) \neq 1$. We summarise this result as follows:

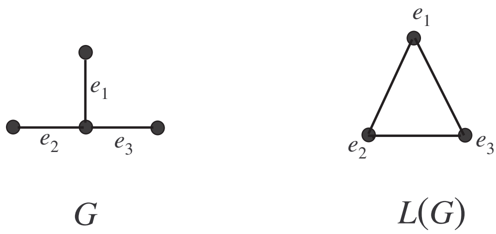

Figure 2: Graph G and its line graph $L(G)$.

Theorem 9. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots v_n\}$, and edge set $E(G) = \{e_1, e_2, \dots e_m\}$. Let $A = (a_{ij})$ be its adjacency matrix. Then G is a T_1 graph if and only if there does not exist an index i such that $\sum_{j=1}^{n} a_{ij} = 1$.*

4. Line Graph and Complement of a Graph

Theorem 10. *The line graph $L(G)$ of a T_1 graph G is T_1.*

Proof. Let e_1 and e_2 be two distinct vertices of $L(G)$. Then e_1 and e_2 are two distinct edges of G. Since $e_1 \neq e_2$, there exist two distinct vertices x and y such that x is incident with e_1 but not incident with e_2 and y is incident with e_2 but not with e_1. Let $e_1 = ux$ and $e_2 = vy$, where u and v need not be distinct. Clearly $x \neq v$ and $u \neq y$. Since G is T_1, there exists an edge f_1 incident with x but not with u. Similarly, there exist an edge f_2 incident with u but not with x. Then $e = e_1 f_2$ is an edge of $L(G)$ incident with e_1 but not with e_2. Similarly we can prove that there exists an edge f incident with e_2 but not with e_1. Therefore, the line graph $L(G)$ of G is T_1. □

Figure 2, shows that line graph of a non-T_1 graph can be T_1. Figure 3, shows that the complement of a T_1 graph in general is not T_1.

Let G be a graph with n vertices. If $\Delta(G) \leq n - 3$, then $\delta(\overline{G}) \geq 2$. Hence we have:

Proposition 11. *If G is a graph with $\Delta(G) \leq n - 3$, where n is the order of G, then $\overline{G}$ is T_1.*

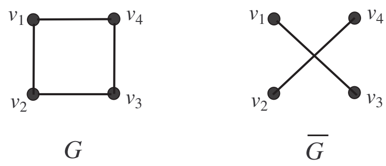

Figure 3: Graph G and its complement $\overline{G}$.

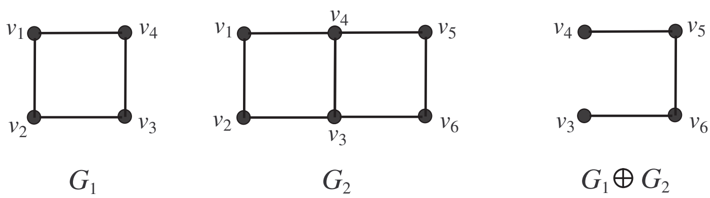

Figure 4: The ring sum of two T_1 graphs.

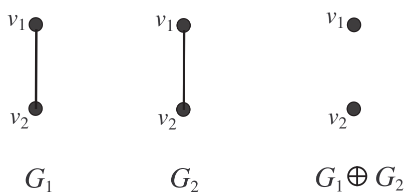

Figure 5: The ring sum of two non-T_1 graphs.

5. Ring Sum and Join

In this section we deal with ring sum and join of two graphs.

Proposition 12. *The ring sum of two graphs with disjoint vertex set is T_1 if and only if both of them are T_1.*

Example 13.

From Figure 4, it follows that, the ring sum of two T_1 graphs need not be T_1 and Figure 5, shows that ring sum of two non-T_1 graphs may be T_1.

Example 14.

Next, we consider the join of two graphs. Figure 6, shows that if $|V(G_1)| \ngeq 2$

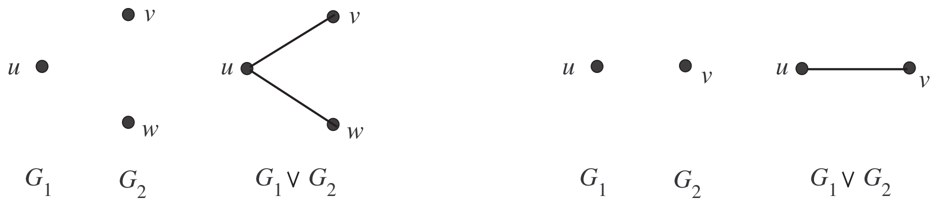

Figure 6: Join of two graphs.

and $|V(G_2)| \ngeq 2$, then $G_1 \vee G_2$ is not T_1.

Theorem 15. *Let G_1 and G_2 be two graphs with $|V(G_1)| \geq 2$ and $|V(G_2)| \geq 2$, then $G_1 \vee G_2$ is T_1.*

Proof. Let u be any vertex of $G_1 \vee G_2$. Without loss of generality we can assume that $u \in V(G_1)$. By the definition of join of graphs, u is adjacent to all the vertices of G_2. Since $|V(G_2)| \geq 2$, $\deg(u) \geq 2$. Since u is arbitrary we get $\delta(G) \geq 2$. Therefore, by Theorem 3, $G_1 \vee G_2$ is T_1. □

6. Corona and Middle Graph

From the definition of corona of two graphs we have:

Theorem 16. *Suppose G_1 is any graph and G_2 is a T_1 graph with no isolated vertices, then $G_1 \circ G_2$ is T_1 . In particular, the corona of two T_1 graphs with no isolated vertices is T_1.*

Proof. Since G_2 is a T_1 graph with no isolated vertices, $|V(G_2)| \geq 3$ and $\delta(G_2) \geq 2$. Therefore, $\delta(G_1 \circ G_2) \geq 2$. Hence by Theorem 3, $G_1 \circ G_2$ is T_1. □

Proposition 17. *If G_1 is any graph and G_2 is a T_1 graph with an isolated vertex, then $G_1 \circ G_2$ can never be T_1.*

Proof. Every isolated vertex of G_2 determines $|V(G_1)|$ pendant edges in $G_1 \circ G_2$. Therefore, $G_1 \circ G_2$ cannot be T_1. □

Remark 18. Figure 7 shows that Theorem 16 need not be true, if we interchange the roles of G_1 and G_2. Also it shows that corona of two T_1 graphs need not be T_1.

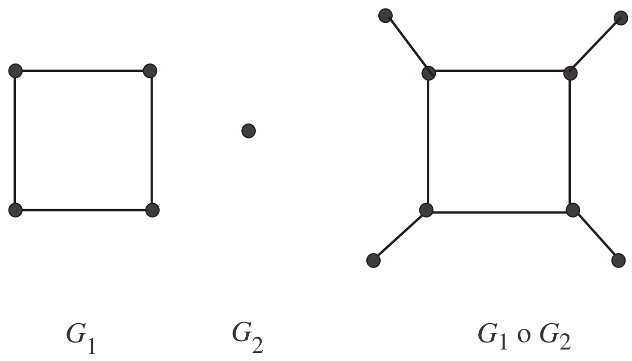

Figure 7: The corona of two graphs.

Another graph that we can derive from the given graph is the middle graph, which also behaves nicely with the T_1 property provided G is a graph with no pendant edges.

Lemma 19. *Let G be a graph with $\delta(G) \geq 2$. Then the middle graph $M(G)$ of G is T_1.*

Proof. Let u and v be two distinct vertices of $M(G)$. As the vertex set of $M(G)$ is $V(G) \cup E(G)$, the following three cases arise.

Case 1. $u, v \in V(G)$
Since $\delta(G) \geq 2$, there exist a vertex w distinct from v such that u is adjacent to w. Let $e = uw$, then ue is an edge of $L(G)$ incident with u but not with v. Similarly we can find an edge f of $L(G)$ incident with v but not with u.

Case 2. $u, v \in E(G)$
Since $u \neq v$, there exist two distinct vertices x and y such that u is incident with x and v is incident with y. Then the edges ux and vy serve the purpose.

Case 3. Suppose $u \in V(G)$ and $v(= e \;\; say) \in E(G)$.
Since $\delta(G) \geq 2$, there exists an edge f distinct from e incident with u. Let $w \neq u$ be an end point of e. Then the edge uf of $M(G)$ is incident with u but not with v and the edge ew is incident with u but not with v.

Hence the lemma □

Theorem 20. *Let G be a graph with no pendant edges, then the middle graph $M(G)$ of G is T_1.*

Proof. We have $V(G) = K \cup H$, where K contains all the isolated vertices of G and H contains all the non isolated vertices of G. We know that the middle graph of an empty graph is empty. Therefore, $M(G) = M(K) \cup M(H)$. Since G is a graph with no pendant edges, $\delta(H) \geq 2$. Therefore, by Lemma 19 $M(H)$ is T_1. Since $M(K)$ is an empty graph it is also T_1. Therefore, $M(G)$ being the union of two T_1 graphs is T_1. Hence the theorem. □

7. Conclusions

In this paper T_1 graphs have been discussed with examples. Sufficient conditions for join of two graphs, middle graph of a graph, corona of two graphs to be T_1 have also been discussed. It was observed that the line graph of a T_1 graph is T_1. Furthermore, the relations of T_1 graph with its incidence matrix and adjacency matrix is discussed.

Acknowledgments

The first author acknowledge the financial support by University Grants Commission of India, under Faculty Development Programme.

References

[1] Rangaswami Balakrishnan and K Ranganathan, *A textbook of graph theory*, Springer Science & Business Media, 2012.

[2] John Adrian Bondy and Uppaluri Siva Ramachandra Murty, *Graph theory with applications*, vol. 290, Macmillan London, 1976.

[3] Gary Chartrand, Linda Lesniak, and Ping Zhang, *Graphs & digraphs*, CRC Press, 2010.

[4] Carmelito E Go and Sergio R Canoy Jr, *Domination in the corona and join of graphs*, International Mathematical Forum, vol. 6, 2011, pp. 763–771.

[5] Frank Harary, *Graph theory*, Addison-Wesley, Reading, MA, 1969.

[6] K Kavitha and NG David, *Dominator chromatic number of middle and total graphs*, International Journal of Computer Applications **49** (2012), no. 20, 42–46.

[7] KR Parthasarathy, *Basic graph theory*, Tata McGraw-Hill, 1994.

[8] Santanu Saha Ray, *Graph theory with algorithms and its applications*, 2013.

[9] Douglas Brent West et al., *Introduction to graph theory*, vol. 2, Prentice hall Upper Saddle River, 2001.