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HAUSDROFF PROPERTY OF CARTESIAN AND TENSOR PRODUCT OF GRAPHS

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Abstract: A simple graph G is said to be Hausdroff if for any two distinct vertices u and v of G, one of the following conditions hold:

1. Both u and v are isolated
2. Either u or v is isolated
3. There exist two nonadjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v.

In this paper we derive sufficient conditions for cartesian and tensor products of two graphs to be Hausdroff.

AMS Subject Classification: 05C76
Key Words: Hausdroff graph, isolated vertex, Cartesian product, end-block, tensor product

1. Introduction

All the graphs considered here are finite and simple. In this paper we denote the set of vertices of G by $V(G)$, the set of edges of G by $E(G)$ and the minimum degree of G by $\delta(G)$.

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The *degree* [4] of a vertex v in a graph G, denoted by $\deg v$, is the number of edges incident with v. A *pendant vertex* [6] in a graph G is a vertex of degree one. A vertex v is *isolated* [2] if $\deg v = 0$. By an *empty graph* [5] we mean a graph with no edges. Two vertices u and v of G are *adjacent*[8], if uv is an edge of G. A simple graph is said to be *complete*[7] if every pair of distinct vertices of G are adjacent in G. A complete graph of n vertices is denoted by K_n. A connected graph that has no cut vertices is called a *block* [5]. A block of G containing exactly one cut vertex of G is called an *end-block* [3] of G. The *Cartesian product*[9] $G\Box H$ of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with vertex set $V(G) \times V(H)$ where the vertex (u_1, v_1) is adjacent to the vertex (u_2, v_2) whenever $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The *Tensor product*(or direct product)[1] $G \times H$ of two graphs G and H is the graph with the vertex set $V(G)\times V(H)$, two vertices (u_1, v_1) and (u_2, v_2) being adjacent in $G \times H$ if, and only if, $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. A graph G is said to be *Hausdroff* [10] if for any two distinct vertices u and v of G, one of the following three conditions hold: (1) Both u and v are isolated (2) Either u or v is isolated (3) There exist two nonadjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v. From the definition of a Hausdroff graph we have if G is a graph with $\delta(G) = 1$, then it cannot be Hausdroff. In particular K_2 is not Hausdroff. Also if G is Hausdroff, then any supergraph of G is Hausdroff.

Theorem 1. *[10] Let $G = (V(G), E(G))$ be a graph with $\delta(G) \geq 3$ then, G is Hausdroff.*

2. Cartesian Product

From the definition of cartesian product of graphs we have:

Proposition 2. *The cartesian product $K_n\Box K_n$ is Hausdroff for every n.*

Proposition 3. *The cartesian product $K_n\Box K_m$ $(n \neq m)$ is Hausdroff if, and only if, either $n = 1$ and $m \geq 4$ or $n \geq 2$ and $m \geq 2$.*

Corollary 4. *The cartesian product $C_n\Box C_m$ is Hausdroff $\forall n, m$.*

Proposition 5. *The cartesian product $P_n\Box P_m$ is Hausdroff $\forall n, m$.*

Example 6.

Example 6 shows that the cartesian product of two non-Hausdroff graphs can be Hausdroff.

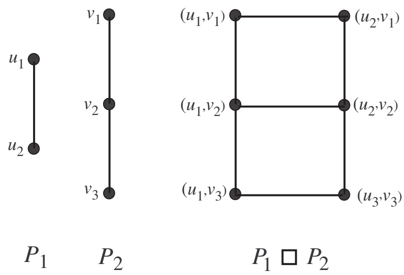

Figure 1: Cartesian product of paths P_1 and P_2

Theorem 7. *Let G_1 and G_2 be two graphs with no isolated vertices. Then $G_1\square G_2$ is Hausdroff.*

Proof. Let (u_i, v_j) and (u_r, v_s) be two distinct vertices of $G_1\square G_2$.

Case 1. $u_i = u_r$

Then the vertices v_j and v_s of G_2 are distinct. Since G_1 is a graph with no isolated vertices, G_1 contains a vertex u_p such that u_i and u_p are adjacent in G_1. Then $(u_i, v_j)(u_p, v_j)$ and $(u_i, v_s)(u_p, v_s)$ are two nonadjacent edges of $G_1\square G_2$.

Case 2. $u_i \neq u_r$

In this case, either $v_j = v_s$ or $v_j \neq v_s$. Suppose $v_j = v_s$. Since $G_1\square G_2 = G_2\square G_1$, the result follows as in Case 1. So we need only to consider the case $v_j \neq v_s$. In this case if u_i is adjacent to u_r, then $(u_i, v_j)(u_r, v_j)$ and $(u_r, v_s)(u_i, v_s)$ are two nonadjacent edges of $G_1\square G_2$ incident with (u_i, v_j) and (u_r, v_s) respectively. If u_i is not adjacent to u_r, since G_1 is free from isolated vertices, there exist vertices u_p and u_q distinct from u_i and u_r such that u_i is adjacent to u_p and u_r is adjacent to u_q in G_1. (u_p may be equal to u_q) Then $(u_i, v_j)(u_p, v_j)$ and $(u_r, v_s)(u_q, v_s)$ are two nonadjacent edges of $G_1\square G_2$ incident with (u_i, v_j) and (u_r, v_s) respectively.

Hence the theorem. □

Remark 8. Theorem 7 need not be true if both the graphs contain isolated vertices. (see. Figure [2]).

If G_1 contains an isolated vertex u and G_2 contains a pendant edge vw then $(u, v)(u, w)$ is a pendant edge of the cartesian product $G_1\square G_2$ of G_1 and G_2. Thus in such cases $G_1\square G_2$ can never be Hausdroff. We state this result as a proposition as follows:

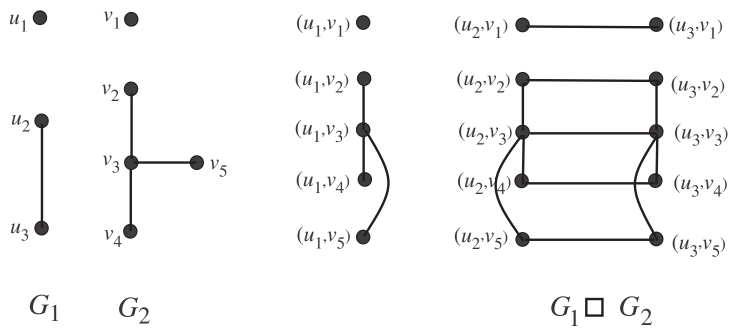

Figure 2: Cartesian product of graphs

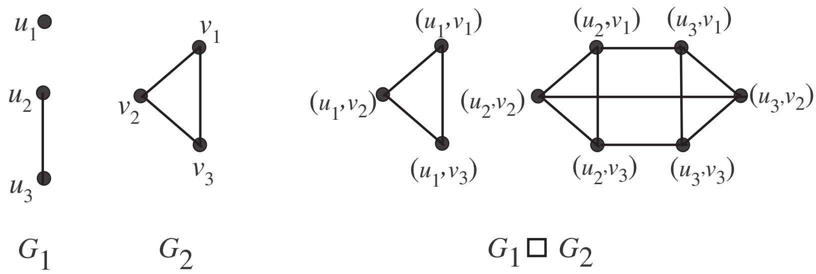

Figure 3: Cartesian product of graphs

Proposition 9. *The cartesian product $G_1 \square G_2$ of two graphs G_1 and G_2 is not Hausdroff if $\delta(G_1) = 0$ and $\delta(G_2) = 1$.*

The question then arise is that what happens to the cartesian product when we increase the minimum degree of the graph G_2. Unfortunately, the result remains failed in certain cases. For example, consider the graphs G_1 and G_2 and their cartesian product in Figure 3. There $\delta(G_1) = 0$ and $\delta(G_2) = 2$. The cartesian product $G_1 \square G_2$ of G_1 and G_2 contains a triangle, hence it cannot be Hausdroff.

Example 10.

But one can overcome this difficult situation by giving some restrictions to the graph G_2.

Theorem 11. *Let G_1 be any graph and G_2 be a graph with no triangle as end-block. If $\delta(G_2) = 2$, then $G_1 \square G_2$ is Hausdroff.*

Proof. Let (u_i, v_j) and (u_r, v_s) be two distinct vertices of $G_1 \Box G_2$.

Case 1. $u_i = u_r$

Suppose v_j and v_s are adjacent vertices of G_2. Since $\delta(G_2) = 2$, there exists a vertex v_p distinct from v_s such that v_j is adjacent to v_p. If v_s and v_p are not adjacent then v_s must be adjacent to some vertex v_q of G_2 distinct from v_j and v_p. Then $(u_i, v_j)(u_i, v_p)$ and $(u_i, v_s)(u_i, v_q)$ are two nonadjacent edges of $G_1 \Box G_2$ incident with (u_i, v_j) and (u_r, v_s) respectively.

If v_s and v_p are adjacent then the subgraph induced by the vertices v_j, v_s, v_p is a triangle in G_2. Since G_2 contains no triangle as end-block either v_s is adjacent to a vertex v_q distinct from v_j and v_p or v_j is adjacent to a vertex v_t distinct from v_s and v_p. In the first case $(u_i, v_j)(u_i, v_p)$ and $(u_i, v_s)(u_i, v_q)$ are two nonadjacent edges of $G_1 \Box G_2$. In the second case $(u_i, v_j)(u_i, v_t)$ and $(u_i, v_s)(u_i, v_p)$ are two nonadjacent edges of $G_1 \Box G_2$.

Suppose v_j and v_s are not adjacent in G_2. In this case since $\delta(G_2) = 2$, we can choose two distinct vertices v_p and v_q of G_2 such that v_p is adjacent to v_j and v_q is adjacent to v_s, then $(u_i, v_j)(u_i, v_p)$ and $(u_i, v_s)(u_i, v_q)$ are two nonadjacent edges of $G_1 \Box G_2$ incident with (u_i, v_j) and (u_r, v_s) respectively.

Case 2. $u_i \neq u_r$

Suppose $v_j = v_s$, since $\delta(G_2) = 2$, there exist two distinct vertices v_p and v_q such that v_j is adjacent to both v_p and v_q. Then $(u_i, v_j)(u_i, v_p)$ and $(u_r, v_s)(u_r, v_q)$ are two nonadjacent edges of $G_1 \Box G_2$.

Suppose $v_j \neq v_s$, then by proceeding as in the proof of Case 1 we get two nonadjacent edges incident with the vertices (u_i, v_j) and (u_r, v_s). First of all we consider the case v_j and v_s are adjacent vertices of G_2. Since $\delta(G_2) = 2$, there exists a vertex v_p distinct from v_s such that v_p is adjacent to v_j. If v_s and v_p are not adjacent then v_s must be adjacent to some vertex v_q of G_2 distinct from v_j and v_p. Which implies $(u_i, v_j)(u_i, v_p)$ and $(u_i, v_s)(u_i, v_q)$ are two nonadjacent edges of $G_1 \Box G_2$ incident with (u_i, v_j) and (u_r, v_s) respectively. If v_s and v_p are adjacent then v_j, v_s, v_p form a triangle. Since G_2 contains no triangle as end-block either v_s is adjacent to a vertex v_q distinct from v_j and v_p or v_j is adjacent to a vertex v_t distinct from v_s and v_p. In the first case $(u_i, v_j)(u_i, v_p)$ and $(u_r, v_s)(u_r, v_q)$ are two nonadjacent edges of $G_1 \Box G_2$. In the second case $(u_i, v_j)(u_i, v_t)$ and $(u_r, v_s)(u_r, v_p)$ are two nonadjacent edges of $G_1 \Box G_2$.

Suppose v_j and v_s are not adjacent in G_2. In this case, since $\delta(G_2) = 2$, we can choose two distinct vertices v_p and v_q of G_2 such that v_j is adjacent to v_p and v_s is adjacent to v_q. Then, $(u_i, v_j)(u_i, v_p)$ and $(u_r, v_s)(u_r, v_q)$ are two nonadjacent edges of $G_1 \Box G_2$.

Thus in all the cases we have proved that for any two distinct vertices

(u_i, v_j), (u_r, v_s) of $G_1 \Box G_2$ there exist two nonadjacent edges e_1 and e_2 of $G_1 \Box G_2$ such that e_1 is incident with (u_i, v_j) and e_2 is incident with (u_r, v_s). Hence $G_1 \Box G_2$ is Hausdroff. □

Theorem 12. *Let G_1 be any graph and G_2 be a graph with $\delta(G_2) \geq 3$, then $G_1 \Box G_2$ is Hausdroff.*

Proof. Let $(u, v) \in V(G_1 \Box G_2)$, then $u \in V(G_1)$ and $v \in V(G_2)$. Since $\delta(G_2) \geq 3$, v is adjacent to at least three vertices, say v_1, v_2, v_3 of G_2. Then the vertex (u, v) is adjacent to the vertices $(u, v_1), (u, v_2)$ and (u, v_3) of $G_1 \Box G_2$. Therefore, $\deg(u, v) \geq 3$. Since (u, v) is an arbitrary vertex of $G_1 \Box G_2$, $\delta(G_1 \Box G_2) \geq 3$. Hence by Theorem 1, $G_1 \Box G_2$ is Hausdroff. □

Proposition 13. *Let G_1 be an empty graph and G_2 be a Hausdroff graph then $G_1 \Box G_2$ is Hausdroff.*

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_m\}$ be the vertex sets of G_1 and G_2 respectively. For $i = 1, 2, \ldots, n$, let $H_i = \langle \{(u_i, v_j); j = 1, 2, \ldots, m\} \rangle$. Then $G_1 \Box G_2 = \bigcup_{i=1}^{n} H_i$. Note that for every i, H_i is isomorphic to G_2 and hence Hausdroff. Therefore, the graph $G_1 \Box G_2$, being the union of Hausdroff graphs, is Hausdroff. □

Remark 14. Since $G_1 \Box G_2 = G_2 \Box G_1$, Propositions 9, 13 and Theorems 7,11,12 are still true even if we interchange the roles of G_1 and G_2.

Theorem 15. *Cartesian product of two Hausdroff graphs is Hausdroff.*

Proof. Let G_1 and G_2 be two Hausdroff graphs. Then, for $i = 1, 2$, $G_i = K_i \cup H_i$, where $V(K_i)$ is the set all isolated vertices of G_i and $V(H_i)$ is the set of all non-isolated vertices of G_i. Hence $G_1 \Box G_2 = (K_1 \Box K_2) \cup (K_1 \Box H_2) \cup (H_1 \Box K_2) \cup (H_1 \Box H_2)$. By Proposition 13, $(K_1 \Box K_2), (K_1 \Box H_2)$ and $(H_1 \Box K_2)$ are Hausdroff. By Theorem 7, $H_1 \Box H_2$ is Hausdroff. Therefore, $G_1 \Box G_2$ is Hausdroff. □

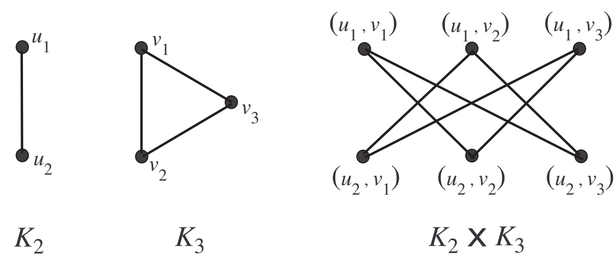

Figure 4: Tensor product of K_2 and K_3

3. Tensor Product

Another interesting graph product that we can consider is that of the tensor product. Let us start with the tensor product of K_2 and K_3. Though both K_2 and K_3 are non-Hausdroff their tensor product seems to be Hausdroff.

Note that the graph K_2 is free from isolated vertices and the graph K_3 has minimum degree 2. Lemma 16 shows that this result is true in general. That is, if $\delta(G_1) \geq 1$ and $\delta(G_2) = 0$, then $G_1 \times G_2$ is Hausdroff.

Lemma 16. *Let G_1 be a graph with no isolated vertices and G_2 be a graph with $\delta(G_2) = 2$. Then $G_1 \times G_2$ is Hausdroff.*

Proof. Let $V(G_1) = \{u_i, i = 1, 2, \ldots m\}$ and $V(G_2) = \{v_j, j = 1, 2, \ldots n\}$. Then, $V(G_1 \times G_2) = \{(u_i, v_j); i = 1, 2, \ldots m, j = 1, 2, \ldots n\}$. Consider two distinct vertices (u_i, v_j) and (u_r, v_s) of $G_1 \times G_2$. Since G_1 contains no isolated vertices, the vertex u_i is adjacent to at least one vertex of G_1.

Case 1. u_i and u_r are adjacent vertices of G_1.

Since $\delta(G_2) = 2$, there exists a vertex v_p distinct from v_s such that v_p is adjacent to v_j. Similarly there exists a vertex v_q distinct from v_j such that v_q is adjacent to v_s. Then $(u_i, v_j)(u_r, v_p)$ and $(u_r, v_s)(u_i, v_q)$ are two nonadjacent edges of $G_1 \times G_2$ incident with (u_i, v_j) and (u_r, v_s) respectively.

Case 2. u_i and u_r are nonadjacent vertices of G_1.

Choose vertices u_p and u_q of G_1 which are adjacent to the vertices u_i and u_r respectively. If v_j and v_s are adjacent in G_2 then, the edges $(u_i, v_j)(u_p, v_s)$ and $(u_r, v_s)(u_q, v_j)$ are two nonadjacent edges of $G_1 \times G_2$ incident with (u_i, v_j) and (u_r, v_s) respectively. Otherwise, since $\delta(G_2) = 2$, we can choose two distinct vertices v_p and v_q of G_2 such that v_p is adjacent to v_j and v_q is adjacent to v_s. Then, the edges $(u_i, v_j)(u_p, v_p)$ and $(u_r, v_s)(u_q, v_q)$ are two nonadjacent edges of $G_1 \times G_2$ incident with (u_i, v_j) and (u_r, v_s) respectively.

□

Theorem 17 shows that the restriction $\delta(G_2) = 2$ on the second graph G_2 can be withdrawn.

Theorem 17. *Let G_1 be a graph with no isolated vertices and G_2 be a graph with $\delta(G_2) \geq 2$. Then, the tensor product $G_1 \times G_2$ of G_1 and G_2 is Hausdroff.*

Proof. If $\delta(G_2) = 2$, then the proof follows from Lemma 16. Now suppose $\delta(G_2) \geq 3$. Let (u, v) be a vertex of $G_1 \times G_2$. Since G_1 is a graph with no isolated vertices, the vertex u is adjacent to at least one vertex say w of G_1. Since $\delta(G_2) \geq 3$, the vertex v is adjacent to at least three vertices say v_1, v_2, v_3 of G_2. Then the vertex (u, v) is adjacent to the vertices $(w, v_1), (w, v_2)$, and (w, v_3) of $G_1 \times G_2$. Therefore, $\deg(u, v) \geq 3$. Since (u, v) is arbitrary, $\delta(G_1 \times G_2) \geq 3$. Hence by Theorem 1, $G_1 \times G_2$ is Hausdroff. □

Lemma 18. *If one of G_1 and G_2 be empty graphs then, $G_1 \times G_2$ is Hausdroff.*

Proof. Let one of G_1 and G_2 be empty graphs then, $G_1 \times G_2$ is an empty graph. This implies $G_1 \times G_2$ is Hausdroff. □

Theorem 19. *Let G_1 be any graph and G_2 be a graph with $\delta(G_2) \geq 2$, then $G_1 \times G_2$ is Hausdroff.*

Proof. We can write $G_1 = K \cup H$, where $V(K)$ is the set all isolated vertices of G_1 and $V(H)$ is the set all non-isolated vertices of G_1. Then $G_1 \times G_2 = (K \times G_2) \cup (H \times G_2)$. Since K is empty by Lemma 18, $K \times G_1$ is Hausdroff. By Theorem 17, $H \times G_2$ is Hausdroff. Therefore, the graph $G_1 \times G_2$, being the union of Hausdroff graphs, is Hausdroff. □

Corollary 20. *For every $n, m \geq 3$, $C_n \times C_m$ is Hausdroff.*

Theorem 21. *Let G_1 and G_2 be two graphs such that both G_1 and G_2 contain at least one pendant vertex. Then $G_1 \times G_2$ can never be Hausdroff.*

Proof. Let u be a pendant vertex with pendant edge ux in G_1 and let v be a pendant vertex with uy as pendant edge in G_2. Then $(u, v)(x, y)$ is a pendant edge in $G_1 \times G_2$. Therefore, $G_1 \times G_2$ is not Hausdroff. □

Theorem 22. *Tensor product of any two Hausdroff graphs is Hausdroff.*

Proof. Let G_1 and G_2 be the given Hausdroff graphs. For $i = 1, 2$, we write $G_i = K_i \cup H_i$, where $V(K_i)$ is the set all isolated vertices of G_i and $V(H_i)$ is the set all non-isolated vertices of G_i. Then $G_1 \times G_2 = (K_1 \times K_2) \cup (K_1 \times H_2) \cup (H_1 \times K_2) \cup (H_1 \times H_2)$. Since both K_1 and K_2 are empty graphs by Lemma 18, $K_1 \times K_2$, $K_1 \times H_2$, $H_1 \times K_2$ are Hausdroff. By Theorem 19, $H_1 \times H_2$ is Hausdroff. Therefore, the graph $G_1 \times G_2$, being the finite union of Hausdroff graphs, is Hausdroff. □

4. Conclusion

In this paper we have discussed conditions under which Cartesian product of two graphs is Hausdroff. It is identified that Cartesian product of two Hausdroff Graphs is Hausdroff. Conditions under which Tensor product of two graphs become Hausdroff have been formulated. There are many unsolved problems in this area which are yet to be settled.

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