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Hausdorff Graphs

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Abstract

A simple graph G is said to be Hausdorff if for any two distinct vertices u and v of G, one of the following conditions hold:

1. Both u and v are isolated
2. Either u or v is isolated
3. There exist two non-adjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v.

In this paper we discuss Hausdorff graphs and some examples of it. This paper also deals with the sufficient conditions for K_{mn}, join of two graphs, middle graph of a graph and corona of two graphs to be Hausdorff. The line graph of a given Hausdorff graph is Hausdorff is proved. Moreover, the relations between Hausdorff graph with its incidence matrix and its adjacency matrix are discussed.

Keywords: Hausdorff graph; empty graph; complete graph; bipartite graph; complement of a graph; union of two graphs; join of two graphs; corona of two graphs; ring sum of two graphs; middle graph of a graph; line graph of a graph; incidence matrix; adjacency matrix.

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1 Introduction

All the graphs considered here are finite and simple. In this paper we denote the set of vertices of G by $V(G)$, the set of edges of G by $E(G)$ and the minimum degree of G by $\delta(G)$.

The *degree* [1] of a vertex v in graph G, denoted by $\deg(v)$, is the number of edges incident with v. A *pendant vertex* [2] in a graph G is a vertex of degree one. A vertex v is *isolated* [1] if $\deg(v) = 0$. By an *empty graph* [3] we mean a graph with no edges. A simple graph is said to be *complete* [4] if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n. A graph is *bipartite* [3] if its vertex set can be partitioned into two subsets, X and Y so that every edge has one end in X and other end in Y; such a partition (X, Y) is called a *bipartition* of the bipartite graph. A simple bipartite graph is *complete* if each vertex of X is adjacent to all vertices of Y. A complete bipartite graph with $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. A graph is *Hamiltonian* [4] if it has a spanning cycle. The *union* [5] of two graphs G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* [6] of two graphs G_1 and G_2 denoted by $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. The *corona* [6] of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where i^{th} vertex of G_1 is adjacent to every vertex in i^{th} copy of G_2. The *ring sum* [7] of two graphs G_1 and G_2, denoted by $G_1 \oplus G_2$, is the graph consisting of the vertex set $V(G_1) \cup V(G_2)$ and of edges that are either in G_1 or G_2, but not in both. The *middle graph* [8] of $G = (V(G), E(G))$ is the graph $M(G) = (V(G) \cup E(G), E'(G))$, where $uv \in E'$ if and only if either u is a vertex of G and v is an edge containing u, or u and v are edges having a vertex in common. The *line graph* [] $L(G)$ of a graph G, is the graph whose vertex set is $E(G)$ and edge set is $\{ef : e, f \in E(G)\ \textit{and}\ e,\ f$ have a vertex in common.$\}$

2 Hausdorff Graphs

In this paper we introduce the concept of Hausdorff graphs.

Definition 2.1. A graph G is said to be Hausdorff if for any two distinct vertices u and v of G, one of the following conditions hold:

1. Both u and v are isolated
2. Either u or v is isolated
3. There exist two non-adjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v.

Note that K_1 is trivially Hausdorff.

It is immediate from the definition of Hausdorff graphs that if,

1. G is a graph with $\delta(G) = 1$, then it cannot be Hausdorff. In particular K_2 is not Hausdorff.
2. G is a graph having at least one end block as triangle then G cannot be Hausdorff.
3. G is Hausdorff, then any super graph of G is Hausdorff.
4. a non-empty graph G of order n is Hausdorff then, $n > 3$.

Why this nomenclature?

Let G be a Hausdorff graph. Then the topology generated by the collection of all two point sets consisting of the end vertices of edges of G and singleton sets consisting of isolated vertices of G is a Hausdorff topology on $V(G)$.

Theorem 2.1. *Let $G = (V(G), E(G))$ be a graph with $\delta(G) \geq 3$, then G is Hausdorff.*

Proof. Let u and v be two distinct vertices of G. Since $\delta(G) \geq 3$, given any two distinct vertices u and v there exists at least two distinct vertices x and y distinct from u and v such that u is adjacent to x and v is adjacent to y. Hence we can separate u and v by two non-adjacent edges. Therefore, G is Hausdorff. □

The simplest example for a non-empty Hausdorff graph is $C_n, n > 3$. When $n = 3$, we have C_3. Since each edge of C_3 is incident with the other two edges of C_3, it cannot be Hausdorff. For $n > 3$, let u and v be two distinct vertices of C_n . In this case we can find two distinct vertices w and x such that u is adjacent to w and v is adjacent to x. Then uw and vx are two non-adjacent edges of C_n. So we have:

Lemma 2.2. *The cycle C_n is Hausdorff if and only if $n > 3$.*

Theorem 2.3. *Any Hamiltonian graph with more than 3 vertices is Hausdorff.*

In fact, all Hamiltonian graphs except C_3 is Hausdorff.

Corollary 2.3.1. *Let G be a graph with $|V(G)| > 3$, if $\deg(v) \geq \frac{|V(G)|}{2}$ for every vertex v of G then, G is Hausdorff.*

Proof. The hypothesis of corollary 2.3.1 implies that G is Hamiltonian [3]. □

Corollary 2.3.2. *A non-Hausdorff graph of order $n > 3$ cannot be Hamiltonian.*

Example 2.1. Example 2.1 shows that the converse of Theorem 2.3 need not be true.

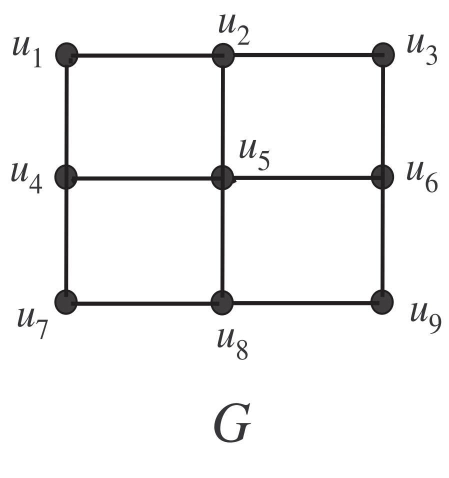

Fig. 1. Non-Hamiltonian Hausdorff graph G

Definition 2.2 ([3])**.** Let G be a graph with order n. The *closure* of G is the graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least n until no such pair remains.

Corollary 2.3.3. *Let G be a graph of order $n \geq 3$. If the closure of a graph G is complete then G is Hausdorff.*

Proof. Let G be a graph of order $n \geq 3$. If the closure of a graph G is complete then G is Hamiltonian [3]. The result follows by Theorem 2.3. □

Since each $K_n, n > 3$ contains a Hamiltonian cycle we have:

Proposition 2.1. *The complete graph K_n is Hausdorff if and only if either $n = 1$ or $n \geq 4$.*

For $n, m > 1$ with $|n - m| \leq 1$, K_{mn} contains a Hamiltonian cycle. Therefore, we have:

Corollary 2.3.4. *For $n, m > 1$ with $|n - m| \leq 1$, K_{mn} is Hausdorff.*

Example 2.2. Fig. 2 is an example of a complete bipartite Hausdorff graph, which is non-Hamiltonian.

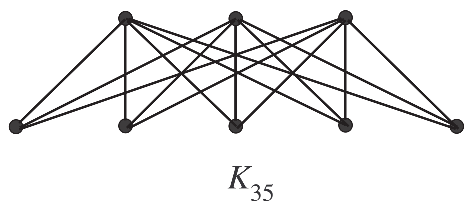

Fig. 2. Non-Hamiltonian K_{mn}

Theorem 2.4. *The complete bipartite graph K_{mn} is Hausdorff if and only if $m, n \geq 2$.*

Proof. If m and n are less than 2, then either $m = 1$, $n = 1$ or $m = 1$, $n = 2$ or $m = 2$, $n = 1$. In the first case K_{mn} is isomorphic to K_2 and in the second and third cases it is isomorphic to P_3, a path with three vertices. Both the graphs are non-Hausdorff. Conversely, suppose $m, n \geq 2$. Let (V_1, V_2) be the bipartition of $V(K_{mn})$. Let u and v be two distinct vertices of K_{mn}.

Case 1. $u, v \in V_1$

If w and x are two distinct vertices of V_2, then uw and vx are two non-adjacent edges of K_{mn}.

Case 2. $u \in V_1$ *and* $v \in V_2$

Let w be a vertex of V_1 other than u and x be a vertex of V_2 other than v. Then ux and vw are two non-adjacent edges of K_{mn}.

The other cases can be proved in a similar manner as in Case 1 or 2. □

Example 2.3. *Ladder graph L_n [9] is Hausdorff if $n \geq 2$.*

Example 2.4. *Prism graph Y_n [10] is Hausdorff for every n.*

Example 2.5. *Petersen graph [3] is Hausdorff.*

Example 2.6. *Helm graph [11] is not Hausdorff.*

3 Incidence Matrix and Adjacency Matrix

Theorem 3.1. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots v_n\}$, and edge set $E(G) = \{e_1, e_2, \dots e_m\}$. Let $M = (m_{ij})$ be its incidence matrix. Suppose that for any two distinct indices i, j there exist two distinct indices k, l $(1 \leq k, l \leq m)$ such that $(m_{ik}, m_{il}) = (1, 0)$ and $(m_{jk}, m_{jl}) = (0, 1)$. In addition if there exist two indices r, s $(1 \leq r, s \leq n)$ such that $(m_{rk}, m_{rl}) = (1, 0)$ and $(m_{sk}, m_{sl}) = (0, 1)$, then G is a Hausdorff graph with non isolated vertices.*

Proof. The hypothesis $(m_{ik}, m_{il}) = (1, 0)$ implies that v_i is incident with e_k and v_i is not incident with e_l. $(m_{jk}, m_{jl}) = (0, 1)$ implies that v_j is incident with e_l and v_j is not incident with e_k. $(m_{rk}, m_{rl}) = (1, 0)$ implies e_k is incident with v_r and e_l is not incident with v_r. $(m_{sk}, m_{sl}) = (0, 1)$ implies e_k is not incident with v_s and e_l is incident with v_s. Thus we get $e_k = v_i v_r$ and $e_l = v_j v_s$. Accordingly, e_k and e_l are two non-adjacent edges of G such that e_l is incident with v_i and e_k incident with v_j. Since v_i and v_j are arbitrary, given any two vertices u and v of G, we can find two non-adjacent edges e_1 and e_2 of G such that e_1 is incident with u and e_2 is incident with v. It follows that G is Hausdorff. □

Remark 3.1. The Example 3.1 shows that the second condition cannot be dropped from Theorem 3.1.

Example 3.1. *Let G be a graph with incidence matrix*

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Here M satisfies only the first condition of Theorem 3.1. The corresponding graph G is given in Fig. 3, which is not Hausdorff. Therefore, both the conditions of Theorem 3.1 are necessary.

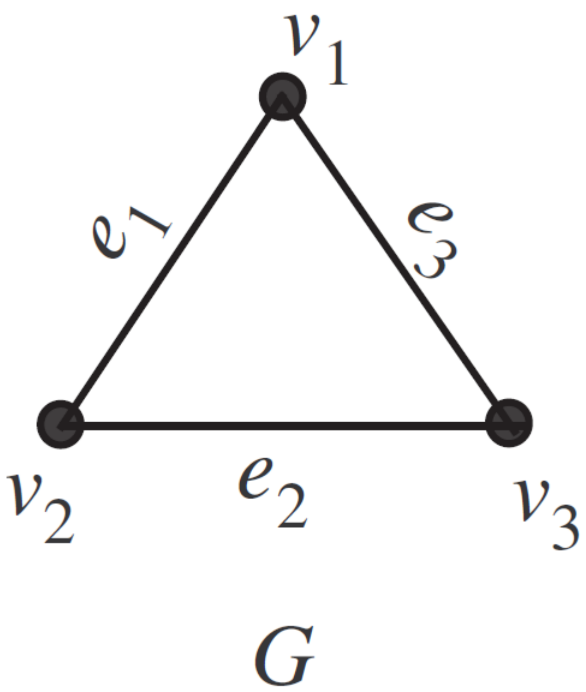

Fig. 3. Graph G with incidence matrix M

Theorem 3.2. *Let G be a Hausdorff graph with no isolated vertices. Let $V(G) = \{v_1, v_2, \dots v_n\}$, $E(G) = \{e_1, e_2, \dots e_m\}$ and $M = (m_{ij})$ be the incidence matrix. Then for any pair of distinct indices (i, j), $1 \leq i, j \leq n$ there exist two distinct indices k, l $(1 \leq k < l \leq m)$ such that $(m_{ik}, m_{il}) = (1, 0)$ and $(m_{jk}, m_{jl}) = (0, 1)$. Moreover, there exist two distinct indices r, s $(1 \leq r < s \leq n)$ such that $(m_{rk}, m_{rl}) = (1, 0)$ and $(m_{sk}, m_{sl}) = (0, 1)$.*

Proof. Let v_i and v_j be two non-isolated vertices of G. Let e_k and e_l be two non-adjacent edges of G such that v_i incident with e_k and v_j incident with e_l. Let v_r and v_s be the other end points of e_k and e_l respectively. Then $(m_{ik}, m_{il}) = (1, 0), (m_{jk}, m_{jl}) = (0, 1), (m_{rk}, m_{rl}) = (1, 0), (m_{sk}, m_{sl}) = (0, 1)$. Hence the theorem. □

We can extend Theorem 3.2 to Hausdorff graphs with isolated vertices as follows:

Theorem 3.3. *Let G be a Hausdorff graph with vertex set $V(G) = \{v_1, v_2, \dots v_n\}$, and edge set $E(G) = \{e_1, e_2, \dots e_m\}$. Let $M = (m_{ij})$ be its incidence matrix. Then give any two distinct non-isolated vertices v_i and v_j there exist two distinct indices k, l $(1 \leq k < l \leq m)$ such that $(m_{ik}, m_{il}) = (1, 0)$ and $(m_{jk}, m_{jl}) = (0, 1)$. Moreover, there exist two distinct indices r, s $(1 \leq r < s \leq n)$ such that $(m_{rk}, m_{rl}) = (1, 0)$ and $(m_{sk}, m_{sl}) = (0, 1)$.*

Theorem 3.4. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots v_n\}$. Let $A = (a_{ij})$ be its adjacency matrix. Suppose for any two distinct indices i, j there exist two distinct indices k, l where, $k \neq j$ and $l \neq i$ such that $a_{ik} = 1,\ a_{jk} = 0$ and $a_{il} = 0,\ a_{jl} = 1$. Then G is Hausdorff.*

Proof. Let v_i and v_j be two distinct vertices of G, then by hypothesis there exist two distinct indices k, l where, $k \neq j$ and $l \neq i$ such that $a_{ik} = 1,\ a_{jk} = 0$ and $a_{il} = 0,\ a_{jl} = 1$. Now $a_{ik} = 1$ implies that v_i and v_k are adjacent vertices of G. $a_{jl} = 1$ implies that v_j and v_l are adjacent vertices of G. Let $e = v_i v_k$ and $f = v_j v_l$. Then e and f are two non-adjacent edges of G. Hence G is Hausdorff. □

Remark 3.2. If G is a connected Hausdorff graph with adjacency matrix $A = (a_{ij})$ then for any two distinct indices i, j there exist two distinct indices k, l $(k \neq j, l \neq i)$ such that $a_{ik} = 1$ and $a_{jl} = 1$

4 Line Graph and Complement of a Graph

Theorem 4.1. *Line graph of a Hausdorff graph is Hausdorff.*

Proof. Let $G = (V(G), E(G))$ be a Hausdorff graph and $L(G)$ be its line graph. Let e_1 and e_2 be two distinct vertices of $L(G)$. Then e_1 and e_2 are two distinct edges of G.

Case 1. *e_1 and e_2 are adjacent edges of G.*

Let $e_1 = ux$ and $e_2 = uy$ for some vertex u of G. Since G is Hausdorff there exist two non-adjacent edges f_1 and f_2 of G such that the vertex x is incident with the edge f_1 and the vertex u is incident with the edge f_2. Suppose $f_2 \neq e_2$, let $f_1 = xp$ and $f_2 = uq$. Then $e_1 f_1$ and $e_2 f_2$ are non-adjacent edges of $L(G)$. Suppose $f_2 = e_2$, since G is Hausdorff, there exist edges g_1 and g_2 of G such that u is incident with g_1 and y is incident with g_2. If $g_1 = e_1$, then $e_1 f_1$ and $e_2 g_2$ are non-adjacent edges of $L(G)$. If $g_1 \neq e_1$, then $e_1 g_1$ and $e_2 g_2$ are non-adjacent edges of $L(G)$.

Case 2. *e_1 and e_2 are non- adjacent edges of G.*

Let $e_1 = uv$ and $e_2 = xy$. Since G is Hausdorff there exist two non-adjacent edges f_1 and f_2 such that u is incident with f_1 and v is incident with f_2. Similarly, there exist two non-adjacent edges g_1 and g_2 such that x is incident with g_1 and y is incident with g_2. If $f_1 = g_1$ then $e_1 f_2$ and $e_2 g_1$ are two non-adjacent edges incident with e_1 and e_2 in $L(G)$ respectively. If $f_1 \neq g_1$ then $e_1 f_1$ and $e_2 g_1$ are two non-adjacent edges incident with e_1 and e_2 in $L(G)$ respectively.

Hence the theorem. □

Example 4.1. Example 4.1 shows that the line graph of a non-Hausdorff graph can be Hausdorff.

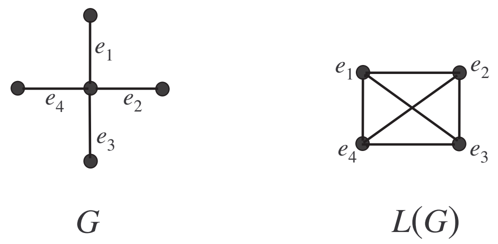

Fig. 4. Graph G and its line graph $L(G)$

Remark 4.1. Though the Hausdorff property behaves well in the case of line graphs it behaves poorly in the case of complement of a graph. Fig. 5 not only substantiate this remark but also conveys that the complement of a non-Hausdorff graph can be Hausdorff.

Example 4.2.

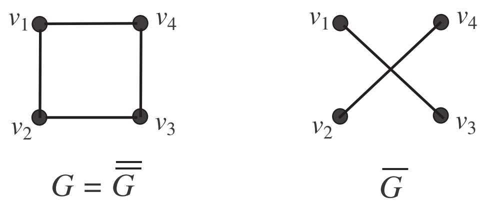

Fig. 5. A graph and its complement

Suppose G is a graph such that for each vertex v of G, there exists at least three vertices of G which are not adjacent to v in G. Then $\delta(\overline{G}) \geq 3$. Hence we have :

Proposition 4.1. *If G is a graph such that for each vertex v of G, there exists at least three vertices of G which are not adjacent to v in G, then $\overline{G}$ is a Hausdorff graph.*

5 Operations on Graph

In this section we deal with union, ring sum and join of two graphs. In general union of two Hausdorff graph is not a Hausdorff graph. However, it is self explanatory that if a finite collection of Hausdorff graphs have pairwise disjoint vertex sets then, their union is Hausdorff.

Proposition 5.1. *Let G be the union of disjoint Hausdorff graphs $G_1, G_2, G_3, \ldots, G_n$. Then, G is Hausdorff.*

As the ring sum of two graphs with disjoint vertex sets is just their union, its immediate from Proposition 5.1 that:

Proposition 5.2. *The ring sum of two graphs with disjoint vertex set is Hausdorff if and only if both of them are Hausdorff.*

Example 5.1. From Fig. 6 it follows that, the ring sum of two Hausdorff graphs need not be Hausdorff.

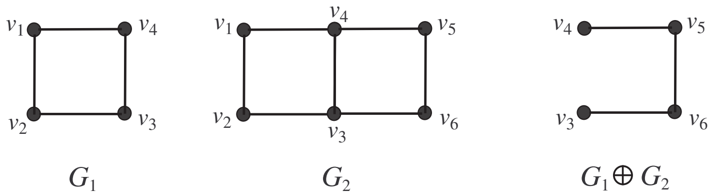

Fig. 6. The ring sum of two graphs

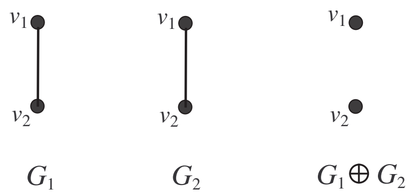

Fig. 7. The Hausdorff ring sum of two non-Hausdorff graphs

Next, we consider the case of join of graphs. Let G_1 and G_2 be two graphs with orders m and n respectively. If $m, n \geq 2$ and $|m - n| \leq 1$, then $G_1 \vee G_2$ is Hamiltonian. Hence we have :

Proposition 5.3. *Let G_1 and G_2 be two graphs with orders m and n respectively, where $m, n \geq 2$, then $G_1 \vee G_2$ is a Hausdorff graph if $|m - n| \leq 1$.*

If $|m - n| > 1$ then $G_1 \vee G_2$ is not Hamiltonian. Even then, the join of any two graphs with no isolated vertices is a Hausdorff graph.

Theorem 5.1. *Let $G_1 = (V(G_1), E(G_1)$ and $G_2 = (V(G_2), E(G_2))$ be two graphs having no isolated vertices then $G_1 \vee G_2$ is Hausdorff.*

Proof. Since G_1 and G_2 are two graphs with no isolated vertices, $\delta(G_1)$ and $\delta(G_2) \geq 1$. Let u and v be two distinct vertices of $G_1 \vee G_2$.

Case 1. *$u \in V(G_1)$ and $v \in V(G_2)$*
Since $\delta(G_1) \geq 1$, u is adjacent to at least one vertex x of G_1. Similarly, v is adjacent to at least one vertex y of G_2. Then ux and vy are two non- adjacent edges of $G_1 \vee G_2$

Case 2. *Both u and v are vertices of G_1*
Let x and y be two distinct vertices of G_2. Then ux and vy are two non-adjacent edges of $G_1 \vee G_2$. The case where both $u, v \in V(G_2)$ is similar.

Thus for any two distinct vertices u, v of $G_1 \vee G_2$ there exist two non-adjacent edges e_1 and e_2 of $G_1 \vee G_2$ such that e_1 is incident with u and e_2 is incident with v. This implies that $G_1 \vee G_2$ is Hausdorff. □

Proposition 5.4. *If G_1 is a graph with an isolated vertex and if no copy of K_1 or K_2 are components of G_2, then $G_1 \vee G_2$ is Hausdorff.*

Proof. Let u and v be two distinct vertices of $G_1 \vee G_2$

Case 1. $u, v \in V(G_2)$

By hypothesis, even if u and v are adjacent in G_2, there exists another vertex x of G_2, which is adjacent to either u or v. Let us suppose that x is adjacent to u in G_2. Let w be an isolated vertex of G_1. Then ux and vw are two non-adjacent edges of $G_1 \vee G_2$.

Case 2. $u \in V(G_1)$ *and* $v \in V(G_2)$

By hypothesis there exist two more vertices x and y in G_2 such that one of them, say x is adjacent to v. Then vx and uy are two non-adjacent edges of $G_1 \vee G_2$.

Case 3. $u, v \in V(G_1)$

Let x and y be two distinct vertices of G_2. Then ux and vy are two non-adjacent edges in $G_1 \vee G_2$.

Thus, for any two distinct vertices $u, v \in V(G_1 \vee G_2)$ there exist two non-adjacent edges e_1 and e_2 of $G_1 \vee G_2$ such that e_1 incident with u and e_2 incident with v. This implies $G_1 \vee G_2$ is Hausdorff. □

Remark 5.1. If G_1 contains an isolated vertex and G_2 contains K_2 as one of its component then $G_1 \vee G_2$ can never be Hausdorff.

Theorem 5.2. *The join $G_1 \vee G_2$ of two empty graphs G_1 and G_2 is Hausdorff if and only if* $|V(G_1)|, |V(G_2)| \geq 2$.

Proof. Suppose $|V(G_1)| = 1$ and $V(G_1) = \{w\}$. Let u and v be two distinct vertices of $G_1 \vee G_2$. Suppose $u, v \in V(G_2)$. Since G_2 is an empty graph and G_1 contains only one vertex w, every edge of $G_1 \vee G_2$ must be incident with w. So it is impossible to find two non-adjacent edges e_1 and e_2 of $G_1 \vee G_2$ such that e_1 is incident with u and e_2 is incident with v. Therefore, $|V(G_1)|, |V(G_2)| \geq$ 2.Conversely, suppose $|V(G_1)|, |V(G_2)| \geq 2$. Let u and v be two distinct vertices of $G_1 \vee G_2$. Suppose $u, v \in V(G_1)$. Let x and y be two distinct vertices of G_2. Then clearly ux and vy are two non adjacent edges of $G_1 \vee G_2$. Similarly, the case $u, v \in V(G_2)$. Now, suppose $u \in V(G_1)$ and $v \in V(G_2)$. Since $|V(G_1)|, |V(G_2)| \geq 2$ we can find a vertex x other than u in $V(G_1)$ and a vertex y other than v in $V(G_2)$. Then clearly uy and vx are two non adjacent edges of $G_1 \vee G_2$. Thus, given any two distinct vertices $u, v \in V(G_1 \vee G_2)$ we find two non- adjacent edges e_1 and e_2 of $G_1 \vee G_2$ such that e_1 incident with u and e_2 incident with v. This implies $G_1 \vee G_2$ is Hausdorff. □

Remark 5.2. Arbitrary join of Hausdorff graphs need not be Hausdorff.

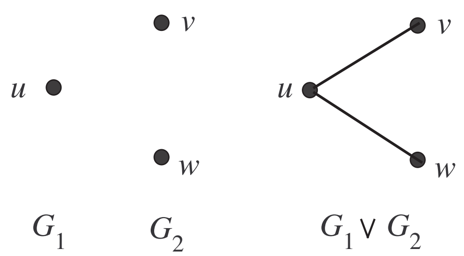

Fig. 8. Graphs G_1 and G_2 and their join

6 Corona and Middle Graph

From the definition of corona of two graphs we have:

Theorem 6.1. *Suppose G_1 is any graph and G_2 is a Hausdorff graph with no isolated vertices, then $G_1 \circ G_2$ is . In particular, the corona of two Hausdorff graphs with no isolated vertices is Hausdorff.*

Proof. Since G_2 is a Hausdorff graph with no isolated vertices, $|V(G_2)| \geq 4$ and $\delta(G_2) \geq 2$, $\delta(G_1 \circ G_2) \geq 3$. Hence by Theorem 2.1, $G_1 \circ G_2$ is Hausdorff. □

Proposition 6.1. *If G_1 is any graph and G_2 is a Hausdorff graph with an isolated vertex, then $G_1 \circ G_2$ can never be Hausdorff.*

Proof. Every isolated vertex of G_2 determines $|V(G_1)|$ pendant edges in $G_1 \circ G_2$. Therefore, $G_1 \circ G_2$ cannot be Hausdorff. □

Remark 6.1. Fig. 9 shows that Theorem 6.1 need not be true if we interchange the roles of G_1 and G_2.

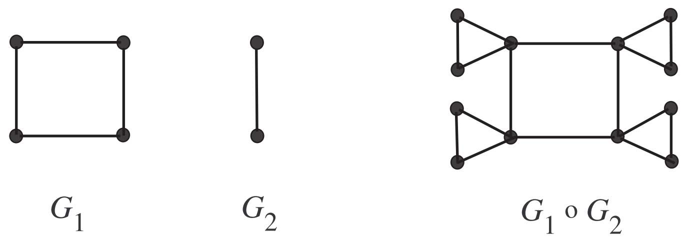

Fig. 9. The corona of two graphs

Corollary 6.1.1. *Corona of two Hausdorff graph need not be Hausdorff.*

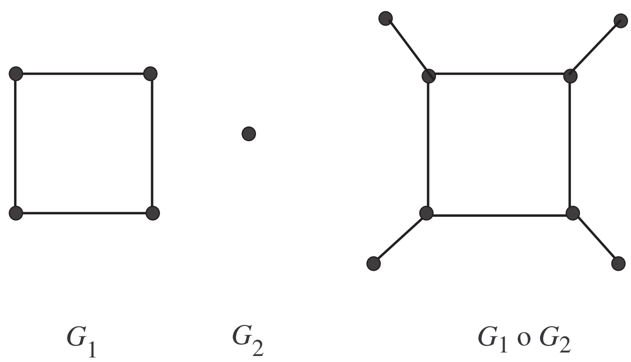

Fig. 10. The corona of two Hausdorff graphs

Another graph that we can derive from the given graph is the middle graph, which behaves nicely with the Hausdorff property provided $\delta(G) \geq 2$.

Theorem 6.2. *Let G be a graph with $\delta(G) \geq 2$. Then the middle graph $M(G)$ of G is Hausdorff.*

Proof. Let u and v be two distinct vertices of $M(G)$. As the vertex set of $M(G)$ is $V(G) \cup E(G)$, the following three cases arise.

Case 1. $u, v \in V(G)$

Subcase 1. *Suppose u and v are adjacent vertices of G.*
Since $\delta(G) \geq 2$, there exists one more vertex w in G which is adjacent to u. Let us denote the edges uv and uw of G by e and f respectively. Then uf and ve are two non-adjacent edges of $M(G)$.

Subcase 2. *Suppose u and v are not adjacent vertices of G.*

Since $\delta(G) \geq 2$, we can find two more distinct vertices x and y distinct from u and v such that u and v are adjacent to x and y respectively in G. Then ue and vf are two non-adjacent edges of $M(G)$ where $e = ux$ and $f = vy$.

Case 2. $u, v \in E(G)$
Since $u \neq v$, there exist two distinct vertices x and y such that u is incident with x and v is incident with y. Then ux and vy are two non-adjacent edges of $M(G)$.

Case 3. *Suppose $u \in V(G)$ and $v(= e \text{ say}) \in E(G)$.*

Subcase 1. *Suppose e is incident with u in G.*

Let $e = uw$ be an edge of G. Since $\delta(G) \geq 2$ there exists an edge f different from e such that f is incident with u. Then uf and ew are two non-adjacent edges of $M(G)$

Subcase 2. *Suppose e is not incident with u in G.*

Since $\delta(G) \geq 2$, there exists an edge f incident with u. Let w be one of the end point of e. Then uf and ew are two non-adjacent edges of $M(G)$.

Thus, for any two distinct vertices $u, v \in V(M(G))$ we find two non- adjacent edges e_1 and e_2 of $M(G)$ such that e_1 is incident with u and e_2 is incident with v. This implies $M(G)$ is Hausdorff. □

7 Conclusions

In this paper Haudroff graphs have been discussed with examples. Sufficient conditions for K_{mn}, join of two graphs, middle graph of a graph, corona of two graphs to be Hausdorff have also been discussed. It was observed that the line graph of a Hausdorff graph is Hausdorff. Furthermore, the relations of Hausdorff graph with its incidence matrix and its adjacency matrix are discussed. More problems in this area of study remain unsolved and subsequently there is a wide scope for further studies.

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Competing Interests

The authors declare that no competing interests exist.

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