19P303 (Pages: 2) Name……………………..

Reg. No…………………..

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2020

(CBCSS-PG)

CC19P MTH3 C13 - FUNCTIONAL ANALYSIS

(Mathematics)

(2019 Admission Regular)

Time: Three Hours Maximum : 30 Weightage

Part A (Short answer questions)

Answer ***all*** questions. Each question has 1 weightage.

1. Define a normed space and give an example.
2. Define a quotient space and prove that two cosets of a linear space either coincide or they are disjoint.
3. If $z \in I[x, y]$, prove that $\|x - y\| = \|x - z\| + \|z - y\|$.
4. State and prove Pythagorean Theorem.
5. If f is a non zero linear functional on a linear space, prove that $codim \ker f = 1$.
6. Prove that $|\langle x, y\rangle| \leq \langle x, x\rangle^{1/2} \cdot \langle y, y\rangle^{1/2}$, for all vectors x, y in a linear space H with inner product$\langle \cdot , \cdot \rangle$.
7. Prove that A is a bounded operator if and only if it is a continuous operator.
8. Show that the subset of a precompact set is precompact.

(8 x 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT I

9. Prove that $\|a + b\|_p \leq \|a\|_p + \|b\|_p$; $1 \leq p \leq \infty$, for every sequence of scalars $a = (a_i), b = (b_i)$.
10. Prove that complement of an open set is closed. Also prove that union of open sets is open.
11. Prove that $C[a, b]$ is a Banach space.

UNIT II

12. Prove that the Hilbert space H is separable if and only if there exists a complete orthonormal system $\{e_i\}_{i\geq 1}$.

13. Prove that for every closed subspace L of H, $L \oplus L^{\perp} = H$ and $(L^{\perp})^{\perp} = L$.

14. State and prove Bessel's inequality. Also prove that for any $x \in H$ and any orthonormal system $\{e_i\}_1^{\infty}$, there exists a $y \in H$ such that $y = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$.

UNIT III

15. Prove or disprove. "*$L(X \mapsto Y)$ is always a Banach space, where X, Y are normed spaces*." Justify your answer.

16. If $A: X \mapsto Y$ is a compact operator, then prove that $A^*: Y^* \mapsto X^*$ is compact.

17. If A is an operator on $L_2[a,b]$ defined by $Af = k(t) \cdot f(t)$, where $k(t)$ is a continuous function on $[a,b]$, then prove that A is a bounded linear operator and $\|A\| \coloneqq M \coloneqq \max_{a \le t \le b} |k(t)|$.

(6 x 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. (a) State and prove completion theorem.

    (b) Prove that l^{∞} is a Banach space.

19. State and prove Hölder's inequality for both the scalar sequences and integrable functions.

20. (a) State and prove Riesz representation theorem.

    (b) Give an example for a non-separable Hilbert space.

21. (a) Prove that M is relatively compact if and only if for every $\epsilon > 0$ there exists a finite ϵ −net in M.

    (b) Prove that strong convergence is weaker than norm convergence.

(2 x 5 = 10 Weightage)

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