19P301 (Pages : 2) Name : ....

Reg. No : ....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2020

(CUCSS-PG)

CC19P MTH3 C11 - MULTIVARIABLE CALCULUS AND GEOMETRY

(Mathematics)

(2019 Admission - Regular)

Time: Three hours Maximum : 30 weightage

PART A

Answer ***all*** questions. Each question carries 1 weightage.

1. (a) Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathbf{x} \in \mathbb{R}^n$. Show that A is differentiable at $\mathbf{x}$.
   (b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + 2y,\ x)$. Find $T'(1, 0)$?
2. Show that $det\ [A] \neq 0$, if A is an invertible linear operator on $\mathbb{R}^n$.
3. (a) Find a parametrization of the level curve $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.
   (b) Find the Cartesian equation of the parametrized curve $\gamma(t) = (e^t,\ t^2)$.
4. Prove that any reparametrization of a regular curve is regular
5. Compute the curvature of the curve $\gamma(t) = \left(\frac{4}{5}\cos t,\ 1 - \sin t,\ -\frac{3}{5}\cos t\right)$.
6. Define ruled surface and give an example for a ruled surface.
7. Find the first fundamental form of the plane $\sigma(u, v) = \mathbf{a} + u\mathbf{p} + v\mathbf{q}$ with $\mathbf{p}$ and $\mathbf{q}$ are perpendicular unit vectors.
8. Define Gaussian and Mean curvatures. **($8 \times 1 = 8$ weightage)**

PART B

Two questions should be answered from each unit. Each question carries 2 weightage.

UNIT I

9. Prove that a linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X.
10. Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x} \in \mathbb{R}^n$. Also show that $\|A\| = |\mathbf{y}|$.
11. (a) Suppose $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and A_x is invertible. Prove that for each $k \in \mathbb{R}^m$ there exists a unique $h \in \mathbb{R}^n$ such that $A(h, k) = 0$.
    (b) State implicit function theorem.

UNIT II

12. Find the torsion of the circular helix $\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$.

13. Show that the unit sphere S^2 is a smooth surface.

14. Suppose S_1 and S_2 are smooth surfaces. Let $f : S_1 \to S_2$ be a local diffeomorphism and let γ be a regular curve on S_1. Show that $f \circ \gamma$ is a regular curve on S_2.

UNIT III

15. Show that the area of a surface patch is unchanged by reparametrization.

16. Compute the second fundamental form of the unit sphere.

17. Find the Gaussian curvature of a surface of revolution $\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$, where $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$.

($6 \times 2 = 12$ Weightage)

PART C

(Answer any two questions. Each question carries 5 weightage.)

18. Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Prove that $f \in \mathscr{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

19. (a) Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$. If $A \in \Omega$, $B \in L(\mathbb{R}^n)$, and $\|B - A\| \cdot \|A^{-1}\| < 1$, then show that $B \in \Omega$.

    (b) State and prove the contraction principle.

20. (a) Prove that a parametrized curve has a unit-speed reparametrization if and only if it is regular.

    (b) Prove that a unit-speed reparametrization of a regular closed curve is closed.

21. (a) Show that the principal curvatures at a point are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point.

    (b) Let S be a connected surface of which every point is an umbilic. Show that S is an open subset of a plane or a sphere.

($2 \times 5 = 10$ Weightage)

****