20P305 (Pages: 2) Name……………………..

Reg. No…………………..

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2021

(CBCSS-PG)

(Regular/Supplementary/Improvement)

CC19P MTH3 C13 - FUNCTIONAL ANALYSIS

(Mathematics)

(2019 Admission onwards)

Time: Three Hours Maximum: 30 Weightage

Part A (Short Answer Type)

Answer ***all*** questions. Each question carries 1 weightage.

1. Define a seminorm and prove that $p(x+y)$ is independent of $y \in E_0$, where p is a seminorm and E_0 is it's kernel.
2. If $z \in \mathrm{I}(x, y)$, prove that $\|x-y\| = \|x-z\| + \|z-y\|$
3. Prove that if $p \geq q \geq 1$, then $l_p \subset l_q$.
4. Define inner product and find $\langle x, y\rangle$, if $x, y \in \mathbb{C}^2; x = (1+i, 1-i), y = (2i-1, i)$.
5. State and prove the parallelogram law.
6. Prove that codim ker $f = 1$, where $f \in E^{\#}\{0\}$.
7. Prove or disprove: "The dual of a normed space is always a Banach space".
8. Prove that strong convergence is weaker than norm convergence.

(8 × 1 = 8 Weightage)

Part B

Answer any ***six*** questions. Each question carries 2 weightage.

UNIT I

9. Prove that two cosets of a linear space are either coincide or disjoint sets. Also prove that the codimension of c_0 inside the space c of convergent sequences is equal to 1.
10. If O is an open set, then prove that the set $F = O^c$is closed. Also prove that if F is a closed set, then the set $O = F^c$ is open.
11. Let E, X_1, X_2 be linear spaces with X_1, X_2 being complete and $T_1: E \to X_1$, $T_2: E \to X_2$ are isometries into dense subspaces of X_1, X_2 respectively. Prove that the natural isometry $T_2 \circ T_1^{-1}: T_1E \to T_2E$ can be extended to an isometry between X_1 and X_2.

UNIT II

12. If $M$is a convex closed set in the Hilbert space H, prove that there exists a unique $y \in M$ such that $\rho(x, M) = \|x-y\|$.

13. Explain Hilbert space with an example. And prove that any two separable infinite dimensional Hilbert spaces are isometrically equivalent .

14. State and prove Bessel's inequality. Also prove that for any $x \in H$ and any orthonormal system$\{e_i\}_1^\infty$, there exists a $y \in H$ such that $y = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$.

UNIT III

15. Let $L \hookrightarrow X$ be a subspace of a normed space X and $x \in X$ such that $dist(x, L) = d > 0$, then prove that there exists $f \in X^*$ such that $\|f\| = 1, f(L) = 0$ and $f(x) = d$.

16. Let X be normed space and Y be a complete normed space. Then prove that $L(X \mapsto Y)$ is a Banach space.

17. Prove that $A : X \mapsto Y$ is compact implies $A^* : Y^* \mapsto X^*$ is compact.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Prove that l_p $1 \le p \le \infty$ is a complete normed space.

19. State and prove Hölder's inequality and Minkowski's inequality for the scalar sequences.

20. (a) State and prove Riesz representation theorem.

    (b) For a normed space X and x_1, x_2 be two elements in X such that $x_1 \neq x_2$, prove that there exists $f \in X^*$ satisfying $f(x_1) \neq f(x_2)$.

21. (a) Prove that M is relatively compact if and only if for every $\epsilon > 0$ there exists a finite $\epsilon -$net in M.

    (b) Prove that $M = \{x(t) \in C[a, b]: |x(t)| < C_1 \; and \; |x'(t)| < C_2\}$ is relatively compact, where C_1, C_2 are positive constants.

(2 × 5 = 10 Weightage)

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