18U601 (Pages: 2) Name: ……………..……….

Reg. No…………..………

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, APRIL 2021

(CUCBCSS-UG)

(Regular/Supplementary/Improvement)

CC15U MAT6 B09 / CC18U MAT6 B09 - REAL ANALYSIS

(Mathematics - Core Course)

(2015 Admission onwards)

Time: Three Hours Maximum: 120 Marks

Section A

Answer ***all*** questions. Each question carries 1 mark.

1. Give an example to show that a continuous function on a set A does not necessarily have an absolute maximum or an absolute minimum on the set.
2. State uniform continuity theorem
3. Define Lipchitz functions. Give an example.
4. Calculate the norm of the partition $\mathcal{P} = (0,1,2,4)$
5. Show that every constant function is Riemann integrable.
6. Find the radius of convergence of the series $1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$
7. Define uniform convergence of a sequence of functions.
8. Find $\lim \left(\frac{x^2+nx}{n}\right)$
9. Show that the integral $\int_1^\infty \frac{1}{\sqrt{x}} dx$ diverges.
10. Investigate the convergence of $\int_0^1 \frac{1}{1-x}\, dx$
11. Show that $\beta(m,n) = \beta(n,m)$
12. State recurrence formula for Gamma function.

(12 × 1 = 12 Marks)

Section B

Answer any ***ten*** questions. Each question carries 4 marks.

13. Let I be a closed bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Show that the set $f(I) = \{f(x): x \in I\}$ is a closed bounded interval.
14. Show that the function $f(x) = \frac{1}{x}$ is not uniformly continuous on $A = \{x \in \mathbb{R}: 0 < x \leq 1\}$
15. Show that every Lipchitz functions on A is uniformly continuous on A
16. If $f(x) = x$ for $x \in [0,1]$, calculate the first few Bernstein polynomials for $f(x)$
17. If $f, g \in \mathcal{R}[a,b]$, show that $f + g \in \mathcal{R}[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$
18. Let $f: [a,b] \to \mathbb{R}$. Show that $f \in \mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$, there exist functions $\alpha_{\varepsilon,}\omega_\varepsilon \in \mathcal{R}[a,b]$ with $\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x)$ for all $x \in [a,b]$ and such that $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$

19. State the substitution theorem of Riemann integration. Use it to evaluate $\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\, dt$
20. Discuss the pointwise convergence of the sequence of functions(x^n)
21. Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f: A \to \mathbb{R}$. Prove that f is continuous on A
22. State and prove Weierstrass M-test.
23. Show that $\int_1^{\infty} \frac{3}{e^x+5}\, dx$ converges.
24. Express $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx$ in terms of beta function.
25. Show that $\beta(m,n) = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$
26. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(10 × 4 = 40 Marks)

Section C

Answer any ***six*** questions. Each question carries 7 marks.

27. State and prove boundedness theorem.
28. Show that $f(x) = \cos x^2, x \in \mathbb{R}$ is not uniformly continuous on $\mathbb{R}$
29. Let $f(x) = x, x \in [0,1]$. Show that $f \in \mathcal{R}[0,1]$
30. If $f: [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, show that $f \in \mathcal{R}[a,b]$
31. State and prove fundamental theorem of calculus first form.
32. Show that if $g_n(x) = \frac{x}{nx+1}$ for $x \geq 0$, then (g_n) converges uniformly on $[0,\infty)$
33. Show that the series $\sum \frac{x}{(nx+1)[(n-1)x+1]}$ is uniformly convergent on any interval $[a,b], 0 < a < b,$ but only point wise on $[0,b]$
34. Test for convergence $\int_{-\infty}^{\infty} \frac{x^3+x^2}{x^6+1} dx$
35. Show that $\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

(6 × 7 = 42 Marks)

Section C

Answer any ***two*** questions. Each question carries 13 marks.

36. a) State and prove location of roots theorem

    b) Is there a real number that is less than its fifth power.
37. a) Let $f: [a,b] \to \mathbb{R}$ and let $c \in (a,b)$. Then $f \in \mathcal{R}[a,b]$ if and only if its restriction to $[a,c]$ and $[c,b]$ are both Riemann integrable. In this case $\int_a^b f = \int_a^c f + \int_c^b f$

    b) Find $F'(x)$ when F is defined on $[0,1]$ by $F(x) = \int_0^{x^2} (1+t^3)^{-1} dt$
38. a) State and prove Cauchy criterion for uniform convergence of sequence of functions.

    b) Prove that $\sqrt{\pi}\,\Gamma(2\mathrm{m}) = 2^{2m-1}\Gamma(m)\,\Gamma(m+\frac{1}{2})$

(2 × 13 = 26 Marks)

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