20P201 (Pages: 2) Name: ……………………

Reg. No: ...………………..

SECOND SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2021

(CUCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH2 C06 - ALGEBRA II

(Mathematics)

(2019 Admission onwards)

Time: Three Hours Maximum:30 Weightage

Part A (Short Answer Questions)

Answer ***all*** questions. Each question carries 1 weightage.

1. If $\mathcal{R}$ is a commutative ring with unity and ${}^{\mathcal{R}}/_{\mathcal{M}}$ is a field, show that $\mathcal{M}$ is a maximal ideal of $\mathcal{R}$.
2. If $\varphi_\alpha: \mathcal{F}[x] \rightarrow$ E be the evaluation homomorphism such that $\varphi_\alpha(a) = a$ for $a \in \mathcal{F}$ and $\varphi_\alpha(x) = \alpha$. Then show that φ_α is a transcendental over $\mathcal{F}$ if and only if it is a one to one map.
3. Find the degree and basis for $Q\left(\sqrt[3]{2}, \sqrt{3}\right)$ over Q.
4. Find the number of primitive 8^{th} roots of unity in $\mathbf{GF(9)}$.
5. Show that the complex zeros of polynomials with real coefficients occur in conjugate pairs.
6. What are the automorphisms of $Q\left(\sqrt{2}, \sqrt{3}\right)$ leaving Q fixed?
7. Find the cyclotomic polynomial $\varphi_5(x)$ over Q.
8. Show that regular 20-gon is constructible.

(8 x 1 = Weightage)

Part B

Answer any ***six*** questions. Each question carries 2 weightage.

Unit 1

9. Show that $\mathcal{F}[x]$ is a principal ideal domain where $\mathcal{F}$ is a field. Is $Q[x]/\langle x^2 - 5x + 6\rangle$ a field. Give reason.
10. If E be the simple extension $\mathcal{F}(\alpha)$ of a field $\mathcal{F}$ and α be algebraic over $\mathcal{F}$, then show That $\beta \in$ E can be uniquely expressed as

    $\beta = b_0 + b_1\alpha + b_2\alpha^2 + \cdots \ldots\ldots\ldots. + b_{n-1}\alpha^{n-1}, b_i \in \mathcal{F}$

    where degree $irr\,(\alpha, \mathcal{F}) = n \geq 1$. Also show that there exists an irreducible polynomial of degree 3 in $Z_3[x]$.

11. Prove that $\mathcal{Q}(\sqrt{3}+\sqrt{7}) = \mathcal{Q}(\sqrt{3},\sqrt{7})$

Unit 2

12. Show that any two fields of same positive prime power order are isomorphic. Also show that if $\mathcal{F}$ is any field, there is an irreducible polynomial in $\mathcal{F}[x]$ of degree n.
13. State and prove Frobenius automorphism. Also prove $\mathcal{F}_{\{p\}}$ is isomorphic to Z_p.
14. Show that $\mathcal{G}(\mathcal{Q}(\sqrt{2},\sqrt{3})/\mathcal{Q})$ has order 4.

Unit 3

15. What is the group of the polynomial $x^3 - 1 \in \mathcal{Q}[x]$ over $\mathcal{Q}$.
16. Find $\phi_3(x)$ and $\phi_8(x)$ over $\mathcal{Z}_2$ and $\mathcal{Z}_3$ respectively.
17. Can the splitting field of $x^{17} - 5$ over $\mathcal{Q}$ has a solvable Galois group? Give reason.

(6 x 2 = 12 Weightage)

Part C

Answer any ***two*** questions from the following four questions.
Each question carries 5 weightage.

18. State and prove Kronecker's Theorem. Also show that $\mathcal{R}[x]/\langle x^2+1\rangle$ is isomorphic to the set $\mathcal{C}$ of complex numbers.
19. State and prove isomorphism extension theorem.
20. Prove that a field E where $\mathcal{F} \leq E \leq \bar{\mathcal{F}}$ is a splitting field over $\mathcal{F}$ iff every automorphism of $\bar{\mathcal{F}}$ leaving $\mathcal{F}$ fixed maps E onto itself and thus induces an automorphism of E leaving $\mathcal{F}$ fixed.
21. State the main theorem of Galois Theory and also prove that for $\mathcal{H} \leq \mathcal{G}(\mathcal{K}/\mathcal{F})$, $\lambda(\mathcal{K}_{\mathcal{H}}) = \mathcal{H}$.

(2 x 5 = 10 Weightage)

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