19P402 (Pages: 2) Name: ……………………….…...

Reg. No……………………..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2021

(CBCSS - PG)

CC19P MTH4 C15 - ADVANCED FUNCTIONAL ANALYSIS

(Mathematics - Core Course)

(2019 Admission - Regular)

Time: Three Hours Maximum: 30 Weightage

Part A (Short Answer questions)

Answer ***all*** questions. Each question has 1 weightage.

1. Explain the residual spectrum with an example.
2. Define a symmetric operator and prove that AB is symmetric if A and B are symmetric operators with $AB = BA$.
3. If P is a projection and $Im\ P \perp \ker P$, then prove that $P = P^*$.
4. Prove that $-I||A|| \leq A \leq I||A||$ for any self-adjoint operator A.
5. State Hilbert theorem on the spectral decomposition of self-adjoint bounded operators.
6. Define perfectly convex set. Prove that $K - x$ is perfectly convex if and only if K is perfectly convex.
7. If A is a linear operator on a Hilbert space H with $DomA = H, A = A^*$, then prove that A is continuous.
8. A linear operator A admits a closure if and only if $Ax_n \to y$ and $x_n \to 0$ imply $y = 0$.

(8 x 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT I

9. For a compact operator T and $\lambda \neq 0$, prove that $\Delta_\lambda = \overline{\Delta_\lambda}$.
10. Prove that $\langle Ax, x\rangle \in \mathbb{R}$ for any $x \in H$ if and only if A is symmetric.
11. State and prove Minimax principle.

UNIT II

12. If $A_0 \leq A_1 \leq \cdots \leq A_n \leq \cdots \leq A$, prove that there exists a strong limit of $(A_n)_n$.
13. If $T: E \to E$ is any linear operator, $Im\ P + \ker P = E$, where P is the projection onto $Im\ P$ parallel to $\ker P$. Then prove that $PT = TP$ if and only if $Im\ P$ and $\ker P$ are invariant subspaces of T.

14. If $Q_n(t)$ and $P_n(t)$ are sequences of polynomials such that $Q_n(t) \searrow \psi(t) \in K$, $P_n(t) \searrow \varphi(t) \in K \; \forall \, t \in [m, M]$ *and* $\psi(t) \leq \varphi(t) \forall \, t \in [m, M]$, prove that

$$\lim_{n\to\infty} Q_n(A) =: B_1 \leq B_2 := \lim_{n\to\infty} P_n(A)$$

UNIT III

15. Prove that every complete metric space M is a set of second category.
16. State and prove Polya's theorem.
17. If μ is perfectly convex on a Banach space X such that $\mu(\lambda x) = \lambda\mu(x)$ for $\lambda \geq 0$, prove that there exists $C > 0$ such that $\mu(x) \leq C\|x\|$ for any $x \in X$.

(6 x 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. State and Prove the first Hilbert-Schmidt theorem.
19. Let A be such that $m \cdot I \leq A \leq M \cdot I$ for some $m, M \in \mathbb{R}$. If P is a polynomial such that $P(z) \geq 0, \forall z \in [m, M]$, prove that $P(z) \geq 0$.
20. Prove that every basis of a Banach space is a Schauder basis.
21. Prove that there exists an equivalent norm $|x|$ on $\mathcal{A}$ such that $|x \cdot y| \leq |x| \cdot |y|$ and $|e| = 1$.

(2 x 5 = 10 Weightage)

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