19P403 (Pages: 2) Name: ....

Reg. No.: ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2021

(CBCSS - PG)

CC19P MTH4 E05 - ADVANCED COMPLEX ANALYSIS

(Mathematics - Elective Course)

(2019 Admission - Regular)

Time: Three Hours Maximum: 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 Weightage.

1. When a set $\mathcal{F} \subseteq C(G, \Omega)$ is said to be equicontinuous over a set $E \subseteq G$?
2. Let d be the metric in $\mathbb{C}_\infty$. Prove that $d\left(\frac{1}{z}, \infty\right) = d(z, 0)$
3. Let $|z| < \frac{1}{2}$. Show that $\frac{1}{2}|z| \leq \log(1+z) \leq \frac{3}{2}|z|$
4. Define the gamma function. Prove that $\Gamma(z+1) = z\Gamma(z)$ for $z \neq 0, -1, -2, ...$
5. If $Re(z) > 1$, prove that $\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty}\left(\int_0^{\infty} e^{-nt}t^{z-1}dt\right)$
6. State Weierstrass factorization theorem.
7. When an entire function is said to have a finite rank?
8. Define *genus* of an entire function.

(8 × 1 = 8 **Weightage)**

Part B

Answer any ***two*** questions from each unit. Each question carries 2 Weightage.

UNIT I

9. State and prove Hurwitz's theorem.
10. Let $\{f_n\}$ be a sequence in $M(G)$ and suppose $f_n \to f$ in $C(G, \mathbb{C}_\infty)$. Prove that either f is meromorphic or $f \equiv \infty$. Also prove that if each f_n is analytic then either f is analytic or $f \equiv \infty$.
11. Let $Re(z_n) > -1$. Prove that the series $\sum \log(1+z_n)$ converges absolutely iff the series $\sum z_n$ converges absolutely.

UNIT II

12. Show that $\sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$ for all $z \in \mathbb{C}$
13. If $Re(z) > 1$, prove that $n^{-z}\Gamma(z) = \int_0^{\infty} e^{-nt}t^{z-1}dt$
14. State and prove Runge's theorem.

Unit III

15. State and prove Mittag-Leffler theorem.

16. State and prove Schwarz reflection principle.

17. Let f be an analytic function on a region containing $\bar{B}(0;r)$ and suppose that $a_1, a_2, ...a_n$ are the zeros of f in $B(0;r)$ repeated according to multiplicity. If $f(0) \neq 0$, prove that

$$\log|f(0)| = -\sum_{k=1}^{\infty} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta$$

($6 \times 2 = 12$ **Weightage)**

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Suppose G is open in $\mathbb{C}$. Prove that there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \cup_{n=1}^{\infty} K_n$ satisfying
    (a) $K_n \subseteq int\ K_{n+1}$
    (b) $K \subseteq G$ and K compact implies $K \subseteq K_n$ for some n.
    (c) Every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$

19. State and prove Arzela-Ascoli theorem.

20. Prove that $\Gamma(z) = \displaystyle\int_0^\infty e^{-t}t^{z-1}dt$ if $Re(z) > 0$.

21. Let f be an entire function of genus μ. Prove that for each positive number α there is a number r_0 such that for $|z| > r_0$

$$|f(z)| < \exp\left(\alpha|z|^{\mu+1}\right)$$

($2 \times 5 = 10$ **Weightage)**

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