19P401 (Pages : 2) Name : ....

Reg. No : ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2021

(CBCSS - PG)

CC19P MTH4 E08 – COMMUTATIVE ALGEBRA

(Mathematics - Elective Course)

(2019 Admission - Regular)

Time : Three Hours Maximum : 30 Weightage

In this question paper, A denotes a commutative ring with 1, S a multiplicatively closed subset of A and M, an A-module.

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Show that in a PID, every non-zero prime ideal is maximal.
2. Is $A[x]$ a finitely generated A-module? Justify.
3. Show that $S^{-1}A$ is the zero ring if and only if $0 \in S$.
4. What are all the primary ideals of $\mathbb{Z}$?
5. When a ring B is said to be integral over A? Give an example.
6. True or False : A subring of a Noetherian ring is Noetherian. Justify.
7. Show that if A is Noetherian, so is $S^{-1}A$.
8. Is $\mathbb{Z}$ an Artin ring? Justify.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

Unit 1

9. If $x \in A$ is a nilpotent, show that $1 + x$ is a unit in A. Deduce that the sum of a nilpotent and a unit is a unit.
10. Show that $A \bigotimes M \cong M$ as A-modules.
11. Show that the sequence $0 \to N' \xrightarrow{u} N \xrightarrow{v} N''$ of A-modules is exact if and only if for all A-modules M, the sequence $0 \to \text{Hom}(M, N') \xrightarrow{\bar{u}} \text{Hom}(M, N) \xrightarrow{\bar{v}} \text{Hom}(M, N'')$ is exact.

Unit 2

12. Show that the prime ideals of $S^{-1}A$ are in one-one correspondence with the prime ideals of A that don't meet S.

13. Let N be a submodule of M. With standard notations, show that the following are equivalent.

    i. $N = M$.
    ii. $N_{\mathfrak{p}} = M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of A.
    iii. $N_{\mathfrak{m}} = M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of A.

14. Let $\mathfrak{q}$ be a primary ideal in A. Show that $r(\mathfrak{q})$ is the smallest prime ideal containing $\mathfrak{q}$.

Unit 3

15. Show that a ring B is integral over A if and only if $S^{-1}B$ is integral over $S^{-1}A$ for all multiplicatively closed subsets S of A.

16. Show that a valuation ring is a local ring and is integrally closed in its field of fractions.

17. Let $0 \to M' \overset{\alpha}{\to} M \overset{\beta}{\to} M'' \to 0$ be an exact sequence of A-modules. Show that M is Noetherian if and only if both M' and M'' are Noetherian.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Show that every non-zero ring has a maximal ideal. Deduce that every non-unit is contained in a maximal ideal.

19. State and prove the existence and uniqueness (universal property) of tensor product.

20. State and prove the Hilbert basis theorem. What about the converse? Justify.

21. Show that the following are equivalent for a subring A of a ring B.

    i. $x \in B$ is integral over A.
    ii. $A[x]$ is a finitely generated A-module.
    iii. $A[x]$ is contained in a subring C of B such that C is a finitely generated A-module.
    iv. There exists a faithful $A[x]$-module M which is finitely generated as an A-module.

(2 × 5 = 10 Weightage)

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