19P404 (Pages: 2) Name: ……………….…...

Reg. No…………………..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2021

(CBCSS - PG)

CC19P MTH4 E09 - DIFFERENTIAL GEOMETRY

(Mathematics - Elective Course)

(2019 Admission - Regular)

Time: Three Hours Maximum: 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Show that the graph of any function $f: \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F: \mathbb{R}^{n+1} \to \mathbb{R}$.
2. Sketch the cylinder over the graph of $f(x) = \sin x$
3. Find the spherical image of the sphere $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ oriented by $(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}$
4. Find the velocity, acceleration and speed of the parametrized curve $\alpha(t) = (\cos t, \sin t, 2\cos t, 2\sin t)$.
5. Show that $\alpha: I \to S$ is geodesic if and only if the covariant acceleration $(\dot{\alpha})'$ is zero along α
6. Find the length of the parametrized curve $\alpha: [-1,1] \to \mathbb{R}^3$ by $\alpha(t) = (\cos 3t, \sin 3t, 4t)$
7. Compute $\int_\alpha x_2 dx_1 - x_1 dx_2$ where $\alpha(t) = (2\cos t, 2\sin t), 0 \le t \le 2\pi$
8. Find the Gaussian curvature of the sphere $x_1^2 + x_2^2 + x_3^2 = 1$

(**8 × 1 = 8 Weightage**)

Part B

Answer any ***two*** questions form each unit. Each question carries 2 weightage.

Unit-I

9. Let U be an open set in $\mathbb{R}^{n+1}$ and $f: U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^\perp$
10. Let $a, b, c \in \mathbb{R}$ be such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$ are of the form $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ where λ_1 and λ_2 are the eigenvalues of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$
11. Let $S \subset \mathbb{R}^{n+1}$ be a connected $n-$ surface in $\mathbb{R}^{n+1}$. Then prove that there exist on S exactly two smooth unit normal fields N_1 and N_2, and $N_2(p) = -N_1(p)$ for all $p \in S$.

Unit-II

12. Let S denote the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$ in $\mathbb{R}^3$. Show that α is geodesic of S if and only if α is of the form $\alpha(t) = (r\cos(at+b),\ r\sin(at+b),\ ct+d)$ for some $a, b, c, d \in \mathbb{R}$

13. Let S be an $n-$ surface in $\mathbb{R}^{n+1}$, let $p, q, \in S$ and let α be a piecewise smooth parametrized curve from p to q. Then prove that the parallel transport $P_\alpha: S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.

14. Prove that the Weingarten map L_p on an n-sphere is a scalar multiplication by a constant.

Unit-III

15. Let V be a finite dimensional real vector space with dot product and $L: V \to V$ be a self adjoint linear operator. Prove that $\exists$ an orthonormal basis for V consisting of eigen vectors of L.

16. Prove that on each compact oriented $n-$ surface S in $\mathbb{R}^{n+1}$ there exist a point p such that the second fundamental form at p is definite.

17. Let S be an $n-$ surface in $\mathbb{R}^{n+1}$and let $f: S \to \mathbb{R}^k$. Prove that f is smooth if and only if $f \circ \varphi: U \to \mathbb{R}^k$ is smooth for each local parametrization $\varphi: U \to S$.

($\mathbf{6 \times 2 = 12}$ **Weightage**)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. a) Let X be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Then prove that there exists an open interval I containg 0 and an integral curve $\alpha: I \to U$ of X such that

    (i) $\alpha(0) = p$

    (ii) If $\beta: \tilde{I} \to U$ is any other integral curve of X with $\beta(0) = p$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$

    b) Find the integral curve through $p = (1,1)$ of the vector field $\boldsymbol{X}$ on $\mathbb{R}^2$ given by $\boldsymbol{X}(p) = (p, X(p))$ Where $X((x_1, x_2)) = (x_2, -x_1)$.

19. Let S be a compact connected oriented $n-$ surface in $\mathbb{R}^{n+1}$exhibited as a level set $f^{-1}(c)$ of a smooth function $f: \mathbb{R}^{n+1} \to \mathbb{R}$ with $\boldsymbol{\nabla f}(p) \neq \mathbf{0}$ for all $p \in S$. Then prove that the Gauss map maps S onto the unit sphere S^n.

20. Let C be an oriented plane curve. Then prove that there exists a global parametrization of C if and only if C is connected.

21. State and prove inverse function theorem for n-surfaces. Hence deduce that if S is a compact, connected oriented n-surface in $\mathbb{R}^{n+1}$ whose Gauss-Kronecker curvature is nowhere zero, then the Gauss map is a diffeomorphism.

($\mathbf{2 \times 5 = 10}$ **Weightage**)

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