20P401 (Pages: 2) Name : ....

Reg. No : ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2022

(CBCSS - PG)

(Regular/Supplemenary/Improvement)

CC19P MTH4 E08 - COMMUTATIVE ALGEBRA

(Mathematics - Elective Course)

(2019 Admission onwards)

Time: 3 Hours Maximum: 30 weightage

PART A

Answer ***all*** questions. Each question carries 1 weightage.

1. Show that Jacobson radical is equal to the nilradical in the ring $A[x]$.
2. State and prove the conditions under which the sequence of A- modules and homomorphisms $0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ become exact.
3. Define tensor product of two modules and show that it is unique up to isomorphism.
4. If S is a multiplicatively closed subset of the ring A, show that the prime ideals of $S^{-1}A$ are in one-to-one correspondence with the prime ideals of A which don't meet S.
5. Prove that the powers of a maximal ideal $\mathbf{m}$ are $\mathbf{m}$- primary.
6. Let A be a subring of a ring B, such that the set $B - A$ is closed under multiplication. Show that A is integrally closed in B.
7. Prove that any homomorphic image of a Noetherian ring is Noetherian.
8. Prove that an Artin ring has only a finite number of maximal ideals.

($8 \times 1 = 8$ Weightage)

PART B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

Unit - 1

9. If $\mathbf{a}$ is a proper ideal of the ring A, prove that there exists a maximal ideal of A containing $\mathbf{a}$.
10. Let $p_1, p_2, ..., p_n$ be prime ideals of a ring A and let $\mathbf{a}$ be an ideal contained in $\bigcup_{i=1}^{n} p_i$. Show that $\mathbf{a} \subseteq p_i$ for some i.
11. If M and N are A- modules, prove that (i) $A \otimes M \cong M$ (ii) $M \otimes N \cong N \otimes M$

Unit - 2

12. Prove that "being injective" and "being surjective" are local properties of A- modules.
13. Let M be an A- module and S a multiplicatively closed subset of the ring A. Prove that $S^{-1}A \otimes_A M \cong S^{-1}M$.
14. If the zero ideal of a ring A is decomposable, prove that the set of all zero divisors of A is the union of prime ideals belonging to 0.

Unit - 3

15. Let $A \subseteq B$ be integral domains, B integral over A. Prove that B is a field if and only if A is a field.

16. Prove that M is a Noetherian A- module if and only if every submodule of M is finitely generated.

17. Prove that in an Artin ring the nilradical η is nilpotent.

($6 \times 2 = 12$ Weightage)

PART C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Let A be a nonzero ring. Prove the following.

    (a) The nilradical η of A is the intersection of all the prime ideals of A.

    (b) $x \in A$ is in the Jacobson radical $\Re$ of A if and only if $1 - xy$ is a unit in $A, \forall y \in A$

19. Let $\mathbf{a}$ be an ideal of the ring A and M be a finitely generated A- module. Let ϕ be a module homomorphism on M such that $\phi(M) \subseteq \mathbf{a}M$. Prove that $\phi^n + a_1\phi^{n-1} + \ldots + a_n = 0$; for $a_i \in \mathbf{a}$. In addition, if $\mathbf{a}$ is contained in the Jacobson radical of A, show that $\mathbf{a}M = M \Rightarrow M = 0$.

20. Let $g : A \to B$ be a ring homomorphism and S a multiplicatively closed subset of S. Assume the following.

    (a) $g(s)$ is a unit in B for all $s \in S$.

    (b) $g(a) = 0 \Rightarrow as = 0$ for some $s \in S$.

    (c) every element of B is of the form $g(a)g(s)^{-1}$ for some $s \in S$ and $a \in A$.

    Prove that there exists a unique isomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$.

21. Prove that every ideal in a Noetherian ring A has a primary decomposition.

($2 \times 5 = 10$ Weightage)

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