22P105 (Pages: 2) Name: ....

Reg.No: ....

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2022

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH1 C05 - NUMBER THEORY

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer any ***all*** questions. Each question carries 1 weightage.

1. Prove that $\forall n \geq 1, \Lambda(n) = -\sum_{d/n} \mu(d) \log d$.
2. Show that $(f^{-1})' = -f' * (f * f)^{-1}$, provided $f(1) \neq 0$.
3. State and prove Legendre's identity.
4. Define Chebyshev's ψ function and show that $\psi(x) = \sum_{m \leq log_2 x} \sum_{p \leq x^{1/m}} \log p$.
5. Let $\{a(n)\}$ be a non-negative sequence such that $\sum_{n \leq x} a(n)[\frac{x}{n}] = x \log x + O(x),\ \forall\ x \geq 1$. Then show that $\sum_{n \leq x} \frac{a(n)}{n} = \log x + O(1),\ \forall x \geq 1$.
6. Using quadratic reciprocity law, determine whether $(219|383) = 1$.
7. Using shift cryptosystem with $N = 27$ and $b = 9$, find the cipher text of 'TOMORROW AT SIX AM'. (blank=26)
8. Find the inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} \pmod{26}$.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT - I

9. (a) Prove that $\forall n \geq 1, \sum_{d/n} \phi(d) = n$.

   (b) Find all integers n such that $\phi(n) = \phi(2n)$.
10. Prove that the set of all multiplicative functions is a subgroup of the set of all arithmetical functions with $f(1) \neq 0$ with respect to the Dirichlet multiplication.

11. Prove that for every $n \geq 1, \sum_{d/n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise} \end{cases}$

UNIT - II

12. Prove that for $x \geq 2, \tau(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt$ and $\pi(x) = \frac{\tau(x)}{\log x} + \int_2^x \frac{\tau(t)}{t\log^2 t}dt.$

13. Show that $\forall\ x \geq 1, \sum_{n\leq x} \psi(\frac{x}{n}) = x\log x - x + O(\log x)$ and $\sum_{n\leq x} \tau(\frac{x}{n}) = x\log x + O(x).$

14. Show that there is a constant A such that $\sum_{p\leq x} \frac{1}{p} = \log\log x + A + O(\frac{1}{\log x}),\ \forall\ x \geq 2.$

UNIT - III

15. If P is an odd integer, then prove that $(-1|P) = (-1)^{\frac{P-1}{2}}$ and $(2|P) = (-1)^{\frac{P^2-1}{8}}$.

16. Solve the system of simultaneous congruence $2x + 3y \equiv 1(\text{mod } 26),\ 7x + 8y \equiv 2(\text{mod } 26).$

17. Describe public key cryptosystem and explain RSA cryptosystem.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Show that the set G of all arithmetical functions f with $f(1) \neq 0$ is an abelian group with respect to the Dirichlet multiplication.

19. State and prove Euler's summation formula. Hence show that

$\forall\ x \geq 1,\ \sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$, if $s > 0, s \neq 1$ where $\zeta$is the Remann zeta functon.

20. Prove that the following relations are logically equivalent:

$(a)\ \lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$

$(b)\ \lim_{x\to\infty} \frac{\tau(x)}{x} = 1$

$(c)\ \lim_{x\to\infty} \frac{\psi(x)}{x} = 1$

21. State and prove Euler's criterion for Legendre's symbol. Also check whether 6 is a quadratic residue modulo 23.

(2 × 5 = 10 Weightage)

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