21P402 (Pages: 2) Name: ……………………….…..

Reg. No: ..…………………..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2023

(CBCSS-PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 C15 – ADVANCED FUNCTIONAL ANALYSIS

(Mathematics)

(2019 Admission onwards)

Time: Three Hours Maximum: 30 Weightage

Part A (Short Answer Questions)

Answer ***all*** questions. Each question carries 1 weightage.

1. Prove that $\sigma(A)$, the spectrum of a bounded operator A, is a closed set.
2. If A is a symmetric operator on a Hilbert space H, then prove that $\sigma_p(A) \subseteq \mathbb{R}$, where $\sigma_p(A)$ denotes the point spectrum of A.
3. Let H be a Hilbert space. Prove that an operator A is symmetric if and only if $\langle A(x), x\rangle \in \mathbb{R}$, for every $x \in H$.
4. If $B \geq 0$, then prove that $11\, B^{11} + 4\, B^4 + 3\, B^3 \geq 0$.
5. Let P be an orthoprojection on a Hilbert space H. Then prove that $0 \leq P \leq I$, where I is the identity operator on H.
6. If P is a projection and $\mathrm{Im}(P) \perp \ker(P)$, then prove that $P = P^*$.
7. Define perfectly convex set and give an example.
8. State true or false and justify: *Closed subspaces of a reflexive normed space are reflexive.*

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question has 2 weightage.

UNIT I

9. Let E be a closed proper subspace of a normed space X. Prove that there exists a $y \in X$ such that $\|y\| = 1$ and distance of y to E, $\mathrm{dist}(y, E) \geq \frac{1}{2}$.
10. Let X be an infinite dimensional Banach space and $T : X \to X$ be a compact operator. Prove that $\sigma(T) = \{0\} \cup \sigma_p(T)$.
11. Let H be a Hilbert space. If A is a symmetric operator on H the prove that $\|A\| = \sup\{\frac{|\langle A(x),\ x\rangle|}{\|x\|^2} : x \in H, x \neq 0\}$.

UNIT II

12. Let H be a Hilbert space and A and B be two non-negative operators such that $AB = BA$. Then prove that AB is non-negative. Can we drop the commutativity here? Justify.
13. Let H be a Hilbert space and let $A_0 \leq A_1 \leq \cdots \leq A_n \leq \cdots \leq A$. Prove that there exists a bounded operator B such that $A_n(x) \to B(x)$, for all $x \in H$.
14. Let $a < m < M < b$and A be a symmetric operator on a Hilbert space H such that $mI \leq A \leq MI$. Construct the spectral integral of A.

UNIT III

15. State and prove closed graph theorem.
16. Let X be a nomred space. If X^* is a separable, then prove that X is separable.
17. Let $\mathcal{H}$ be a Banach algebra. Prove that for every proper ideal $I \subseteq \mathcal{H}$, there exists a maximal ideal M such that $I \subseteq M$.

(6 × 2 = 12 Weightage)

Part B

Answer any ***two*** questions from the following questions. Each question has 5 weightage.

18. Let T be a compact operator on a Banach space X and let $\lambda \neq 0$. Prove that $\text{Im}(\lambda I - T)$ is a closed subspace of X.
19. Let H be a Hilbert space and $A \geq 0$. Prove that there exists an operator $X \geq 0$ such that $X^2 = A$.
20. State and prove the Banach open mapping theorem.
21. Let $\mathcal{H}$ be a Banach algebra. Prove that there exists an equivalent norm $|\cdot|$ on $\mathcal{H}$ such that $|x \cdot y| \leq |x|\,|y|$, for all $x, y \in \mathcal{H}$.

(2 × 5 = 10 Weightage)

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