21P403 (Pages: 2) Name: ....

Reg.No: ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2023

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 E05 - ADVANCED COMPLEX ANALYSIS

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer any ***all*** questions from each unit. Each question carries 1 weightage.

1. Suppose $\mathcal{F} \subseteq (C(G,\Omega))$ is equicontinuous at each point of G. Prove that $\mathcal{F}$ is equicontinuous over each compact subset of G.
2. If d is the metric of $\mathbb{C}_\infty$, show that $d\left(\frac{1}{z},\infty\right) = d(z,0)$ for $z \in \mathbb{C}$.
3. Show that Conformal equivalence is an equivalence.
4. Define the elementary factor function $E_p(z)$. Prove that $E_p(z) \approx 1$ for large p.
5. Define the gamma function. Show that the residue of the gamma function Γ at simple pole $-n$ is given by $Res(\Gamma,-n) = \dfrac{(-1)^n}{n!}$
6. Let $S = \{z : Re(z) \leq A\}$ where $-\infty < A < \infty$. Prove that for $\varepsilon > 0$ there is $\kappa > 1$ such that for all z in S, $\left|\int_\alpha^\beta (e^t-1)^{-1}t^{z-1}dt\right| < \varepsilon$ whenever $\beta > \alpha > \kappa$.
7. Show that the coefficients a_{-n} in the Laurent series for $(e^z-1)^{-1}$ are zeros for $-n \leq -2$
8. When we can consider (f_1, D_1) as an analytic continuation of (f_0, D_0) along a path γ?

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. Suppose G is open in $\mathbb{C}$. Prove that there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \cup_{n=1}^{\infty} K_n$ satisfying

   (i) $K_n \subseteq \text{int } K_{n+1}$ (ii) $K \subseteq G$ and K compact implies $K \subseteq K_n$ for some n.
10. Show that $\mathcal{F} \subseteq (C(G,\Omega)$ is normal iff for every compact set $K \subseteq G$ and $\delta > 0$ there are functions $f_1, f_2, \ldots, f_n \in \mathcal{F}$ such that for $f \in \mathcal{F}$, there is at least one $k,\ 1 \leq k \leq n$ with sup $\{d(f(z), f_k(z)) : z \in K\} < \delta$.

11. Let $\{f_n\}$ is a sequence in $H(G)$ and $f \in (C(G,\mathbb{C}))$ such that $f_n \to f$. Prove that f is analytic and $f_n^{(k)} \to f^{(k)}$ for each integer $k \geq 1$.

UNIT - II

12. Find a factorization for $\cos\left(\frac{\pi z}{4}\right) - \sin\left(\frac{\pi z}{4}\right)$

13. Prove that $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ for $Re(z) > 0$.

14. Prove that $\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin\left(\frac{\pi}{2}z\right)$ for $-1 < Re(z) < 0$.

UNIT - III

15. Let G be a region and let $\{a_k\} \subseteq G$ be a sequence of distinct points such that $\{a_k\}$ has no limit points. For each $k \in \mathbb{N}$, let $S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z-a_k)^j}$ where $m_k \in \mathbb{N}$, $A_{jk} \in \mathbb{C}$. Prove that there exist $f \in M(G)$ whose poles are exactly $\{a_k\}$ and the singular part of *f* at $z = a_k$ is $S_k(z)$.

16. Let *f* be an analytic function on a region containing $\overline{B}(0;r)$ and suppose that $a_1, a_2, \ldots, a_n$ are the zeros of f in $B(0;r)$ repeated according to multiplicity. Let $f(0) \neq 0$, prove that $\log|f(0)| = -\sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta}))d\theta$

17. Prove that if *f* is an entire function of order λ then f' also has order λ.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. (a) Let $Re(z_n) > 0, \forall n \in \mathbb{N}$. Prove that $\prod_{n=1}^{\infty} z_n$ converges to a nonzero number iff the series $\sum_{n=1}^{\infty} \log z_n$ converges.

(b) Let $Re(z_n) > -1$. Prove that the series $\sum \log(1+z_n)$ converges absolutely iff the series $\sum z_n$ converges absolutely.

19. Let (X_n, d_n) are metric spaces for each *n*. Prove that the space $\left(\prod_{n=1}^{\infty} X_n, d\right)$ where $d = \sum_{n=1}^{\infty}\left[\left(\frac{1}{2}\right)^n\left(\frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}\right)\right]$ is a metric space. Also if $\xi^k = \{x_n^k\}_{n=1}^{\infty}$ is in $X = \prod_{n=1}^{\infty} X_n$, then prove that $\xi^k \to \xi = \{x_n\}$ iff $x_n^k \to x_n$ for each n. If each (x_n, d_n) is compact then *X* is compact.

20. (a) State and prove Bohr-Mollerup theorem.

(b) Let K be a compact subset of $\mathbb{C}$ and let E be a subset of $\mathbb{C}_\infty - K$ that meets each component of $\mathbb{C}_\infty - K$. If f is analytic on an open set containing K and $\varepsilon > 0$. Prove that there is a rational function $R(z)$ whose only poles lie in E and $|f(z) - R(z)| < \varepsilon$ for all *z* in K.

21. Let f be an entire function of genus μ. Prove that for each positive number α there is a number r_0 such that for $|z| > r_0 |f(z)| < \exp\left(\alpha|z|^{\mu+1}\right)$

(2 × 5 = 10 Weightage)

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