23P105 (Pages: 2) Name: ....

Reg.No: ....

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2023

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH1 C05 - NUMBER THEORY

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Prove that the Dirichlet product is commutative and associative.
2. Prove that $\forall\ n \geq 1, \sum_{d/n} \Lambda(d) = \log n$.
3. Show that $(f^{-1})' = -f' * (f * f)^{-1}$, provided $f(1) \neq 0$.
4. Define Chebyshev's ψ function and show that $\psi(x) = \sum_{m \leq log_2 x} \sum_{p \leq x^{1/m}} \log p$.
5. Let $\{a(n)\}$ be a non-negative sequence such that $\sum_{n \leq x} a(n)[\frac{x}{n}] = x \log x + O(x),\ \forall\ x \geq 1$. Then show that $\sum_{n \leq x} \frac{a(n)}{n} = \log x + O(1),\ \forall x \geq 1$.
6. Check whether 3 is a quadratic residue modulo 23.
7. Find the cipher text of ‘JANUARY’ in the affine cryptosystem with enciphering key (7, 3) in the 26-letter alphabet system.
8. Find the inverse of the matrix $\begin{bmatrix} 15 & 17 \\ 4 & 9 \end{bmatrix}$ (mod 26)

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT - I

9. (a) Prove that $\forall n \geq 1, \sum_{d/n} \phi(d) = n$.

   (b) Find all integers n such that $\phi(n) = \phi(2n)$.
10. Prove that the set of all multiplicative functions is a subgroup of the set of all arithmetical functions with $f(1) \neq 0$ with respect to the Dirichlet multiplication.

11. Prove that for $x \geq 2, \sum_{p \leq x} [\frac{x}{p}] \log p = x \log x + O(x)$

UNIT - II

12. State and prove Abel's identity.

13. $\forall\, x \geq 1$, Prove the following:

(a) $\sum_{n \leq x} \psi(\frac{x}{n}) = x \log x - x + O(\log x)$

(b) $\sum_{n \leq x} \tau(\frac{x}{n}) = x \log x + O(x)$

14. Show that there is a constant A such that $\sum_{p \leq x} \frac{1}{p} = \log \log x + A + O(\frac{1}{\log x}), \ \forall\, x \geq 2.$

UNIT - III

15. Prove that if P is an odd integer, then $(-1|P) = (-1)^{\frac{P-1}{2}}$ and $(2|P) = (-1)^{\frac{P^2-1}{8}}$.

16. Solve the system: $x + 3y \equiv 1 (\text{mod } 26)$
$7x + 9y \equiv 1 (\text{mod } 26)$

17. (i) Describe about RSA cryptosystem.

(ii) How to send a digital signature in RSA cryptosystem?

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Prove that λ is completely multiplicative and $\forall n \geq 1, \sum_{d/n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise} \end{cases}$. Also show that $\lambda^{-1}(n) = |\mu(n)|$.

19. State and prove Euler's summation formula. Hence show that
$\forall\, x \geq 1, \sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$, if $s > 0, s \neq 1$ where $\zeta$is the Remann zeta functon.

20. Prove that the following relations are logically equivalent:

(a) $\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1$

(b) $\lim_{x \to \infty} \frac{\tau(x)}{x} = 1$

(c) $\lim_{x \to \infty} \frac{\psi(x)}{x} = 1$

21. State and prove quadratic reciprocity law for Legendre's symbol and hence determine whether 219 is a quadratic residue modulo 383.

(2 × 5 = 10 Weightage)

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