22P402 (Pages: 2) Name: ....

Reg.No: ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2024

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 C15 - ADVANCED FUNCTIONAL ANALYSIS

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Show that the set of regular points is an open set .
2. If λ_1 and λ_2 are two distinct elements in point spectrum such that $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$ then prove that x_1 and x_2 are orthogonal.
3. State second Hilbert Schmidt Theorem.
4. Prove that if A is symmetric, then $A^{2n} \geq 0$.
5. If $(A \geq 0)$ and $(\langle Ax, x\rangle = 0)$, then show that $(Ax = 0)$
6. Let $P_1P_2 = P_2P_1 = P$. Then prove that P is an orthoprojection and $E = \operatorname{Im} P = E_1 \cap E_2\, (E_i = P_i H)$..
7. Let $A \in L(X \mapsto Y)$ be onto and one-to-one. Then show that there exists $A^{-1} \in L(X \mapsto Y)$.
8. With example define Banach Algebra.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. Show that if $\dim X = \infty$, then the identity operator $I : X \mapsto X$ is not compact.
10. State and Prove Fredholm's second theorem.
11. Let A be a symmetric operator and let $||A|| = \mu = \sup\{|\langle Ax, x\rangle| : ||x|| = 1\}$. Then prove that either μ or $-\mu$ is in $\sigma(A)$.

UNIT - II

12. Let A be such that $m \cdot I \leq A \leq M \cdot I$ for some $m, M \in \mathbb{R}$ and let P be a polynomial satisfying $P(z) \geq 0$ for all $z \in [m, M]$. Then prove that $P(A) \geq 0$.

13. Let $Q_n(t)$ and $P_n(t)$ be sequences of polynomials. Assume that for all $t \in [m, M]$, $Q_n(t) \searrow \psi(t) \in K$ and $P_n(t) \searrow \varphi(t) \in K$. Let $\psi(t) \leq \varphi(t)$ for all $t \in [m, M]$. Then prove that $\lim_{n\to\infty} Q_n(A) =: B_1 \leq B_2 := \lim_{n\to\infty} P_n(A)$.

14. State Hilbert Theorem.

UNIT - III

15. State and prove Baire- Category theorem.

16. State and Prove Polya theorem.

17. Let $\mathcal{A}$ is a Banach Algebra. Prove that an element x is invertible if and only if x does not belong to any poper ideal.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. State and Prove First Hilbert Schmidt theorem.

19. (i) Let $E_1 = \operatorname{Im} P$ and $E_2 = \ker P$. Then, prove that $E_1 + E_2 = E$ and $E_1 \cap E_2 = 0$ (i.e., $E_1 + E_2$ is a direct sum and $E$is a direct decomposition on E_1 andE_2).

    (ii) Let $T : E \mapsto E$ be any linear operator, $E_1 + E_2 = E$ and let P be the projection onto E_1 parallel to E_2 Then show that $PT = TP$ if and only if E_1 and E_2 are invariant subspaces of T

20. State and prove closed graph theorem.

21. State and prove Banach-Steinhaus theorem.

(2 × 5 = 10 Weightage)

****