22P403 (Pages: 2) Name: ....

Reg.No: ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2024

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 E05 - ADVANCED COMPLEX ANALYSIS

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Prove that $C(G, \Omega)$ is a metric space.
2. Let $\{f_n\}$ is a sequence in $H(G)$ and $f \in (C(G, \mathbb{C}))$ such that $f_n \to f$. Prove that f is analytic.
3. If *d* is the metric of $\mathbb{C}_\infty$, show that $d(z_1, z_2) = d\left(\frac{1}{z_1}, \frac{1}{z_2}\right)$ for $z_1, z_2 \in \mathbb{C}$.
4. When a region G_1 is said to be conformally equivalent to G_2? Show that Conformal equivalence is an equivalence.
5. Prove that the Euler constant of gamma function is given by $\gamma = \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) - \log n\right]$
6. Define Riemann zeta function $\zeta(z)$. If $Re(z) > 1$, then prove that $\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty}\left(\int_0^\infty e^{-nt}t^{z-1}dt\right)$
7. Find the residue of $\frac{1}{e^z-1}$ at $z = 0$
8. With suitable assumptions write Poisson-Jenson formula.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. When a set $\mathcal{F} \in H(G)$ is said to be locally bounded? Show that $\mathcal{F} \in H(G)$ is normal implies it is locally bounded.
10. Suppose $\mathcal{F} \subseteq (C(G, \Omega))$ is normal. Prove that for each $z \in G$, $\{f(z) : f \in \mathcal{F}\}$ has compact closure in Ω and $\mathcal{F}$ is equicontinuous at each point of *G*.
11. Let $\{a_n\}$ be a sequence in $\mathbb{C}$ such that $\lim |a_n| = \infty$ and $a_n \neq 0$ for all $n \geq 1$. Suppose that no complex number is repeated in the sequence an infinite number of times. Let $\{\rho_n\}$ is any sequence of integers such that $\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{\rho_n+1} < \infty$ for all $r > 0$. Prove that $f(z) = \prod_{n=1}^{\infty} E_{\rho_n}(z/a_n)$ converges in $H(\mathbb{C})$. Also prove that the function *f* is an entire function with zeros only at the points a_n. Again if z_0 occurs in the sequence $\{a_n\}$ exactly *m* times, show that *f* has a zero at $z = z_0$ of multiplicity *m*.

UNIT - II

12. Show that $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ for all $z \in \mathbb{C}$.

13. Let K be a compact subset of $\mathbb{C}$ and let E be a subset of $\mathbb{C}_\infty - K$ that meets each component of $\mathbb{C}_\infty - K$. If *f* is analytic on an open set containing K and $\varepsilon > 0$. Prove that there is a rational function $R(z)$ whose only poles lie in E and $|f(z) - R(z)| < \varepsilon$ for all z in K.

14. Show that $\int_0^\infty \cos(t^2) dt = \frac{1}{2}\sqrt{\frac{1}{2}\pi}$

UNIT - III

15. Let G be a region and let $\{a_k\} \subseteq G$ be a sequence of distinct points such that $\{a_k\}$ has no limit points. For each $k \in \mathbb{N}$, let $S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z - a_k)^j}$ where $m_k \in \mathbb{N},\ A_{jk} \in \mathbb{C}$. Prove that there exist $f \in M(G)$ whose poles are exactly $\{a_k\}$.

16. Derive the Jensen's formula.

17. Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. For $0 \leq t \leq 1$, let $R(t)$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. Prove that either $R(t) = \infty$ or $R : [0, 1] \to (0, \infty)$ is continuous.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. (a) Let $Re(z_n) > -1$. Prove that the series $\sum \log(1 + z_n)$ converges absolutely iff the series $\sum z_n$ converges absolutely.

    (b) Let $Re(z_n) > 0$. Prove that the product $\prod_{n=1}^{\infty} z_n$ converges absolutely iff $\sum_{n=1}^{\infty} (z_n - 1)$ converges absolutely.

19. Let (X_n, d_n) are metric spaces for each *n*. Prove that the space $\left(\prod_{n=1}^{\infty} X_n, d\right)$ where $d = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^n \left(\frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}\right)\right]$ is a metric space. Also if $\xi^k = \{x_n^k\}_{n=1}^{\infty}$ is in $X = \prod_{n=1}^{\infty} X_n$, then prove that $\xi^k \to \xi = \{x_n\}$ iff $x_n^k \to x_n$ for each n. If each (x_n, d_n) is compact then *X* is compact.

20. Prove that (i) $\left(1 - \frac{t}{n}\right)^n \leq e^{-t}$ for $t \geq 0$ and $n \geq t$. (ii) $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ for $Re(z) > 0$.

21. (a) Let f_1 and f_2 be entire functions of finite order λ_1, λ_2. Show that $f = f_1 f_2$ has finite order $\lambda \leq max(\lambda_1, \lambda_2)$

    (b) Prove that if *f* is an entire function of order λ then f' also has order λ.

(2 × 5 = 10 Weightage)

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