22P401 (Pages: 2) Name: ....

Reg.No: ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2024

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 E08 - COMMUTATIVE ALGEBRA

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Prove that the only ideals of a field A are 0 and A.
2. Prove that $r(a) = (1)$ if and only if $a = (1)$, where $r(a)$ denotes the radical of a.
3. Let $f : A \to B$ be a ring homomorphism and a is a ideal of A. Then prove that $a \subseteq a^{ec}$.
4. Let M be an A-module. If $M = 0$, then prove that $M_p = S^{-1}M = 0$, for all prime ideal p of A (Here $S = A - p$).
5. Prove that all prime are primary, but every primary ideal need not be prime.
6. Prove that Z (the set of integers) is integrally closed in Q (the set of rational numbers).
7. Prove that the length $l(M)$ is an additive function on the class of all A-modules of finite length.
8. Prove that in a Noetherian ring, nilradical is nil-potent.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. Prove that there is a natural isomorphism $HOM(A, M) \cong M$, for any A-module M.
10. If M and N are two A-modules, prove that $Ann(M + N) = Ann(M) \bigcap Ann(N)$.
11. Let M be a finitely generated A-module. Let a be an ideal of A and let ϕ be an A-module endomorphism of M such that $\phi(M) \subseteq aM$. Then prove that ϕ satisfies an equation of the form $\phi^n + a_1\phi^{n-1} + \ldots + a_n = 0$, where $a_i \in a$.

UNIT - II

12. Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B, for all $s \in S$. Then prove that there exists a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$, where $f : A \to S^{-1}A$ defined by $f(x) = x/1$.

13. If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact at M, then prove that $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact at $S^{-1}M$.

14. If N, P are sub modules of an A-module M and if P is finitely generated, then prove that $S^{-1}(N : P) = (S^{-1}N : S^{-1}P)$.

UNIT - III

15. Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be an exact sequence of A-modules. Then prove that M is Noetherian if and only if M', M'' are Noetherian.

16. Prove that if A is Noetherian, so is $A[x_1, x_2, \ldots, x_n]$.

17. Prove that in an Artin ring A, every prime ideal is maximal.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. (i) Prove that every ring $A \neq 0$ has atleast one maximal ideal.
    (ii) Let A be a ring and m, a maximal ideal of A such that every element of $1 + m$ is a unit in A. Then prove that A is a local ring.

19. (i) Define tensor product of two A-modules M and N.
    (ii) Prove that the tensor product of two A-modules M and N exist and is unique upto isomorphism.

20. (i) Let a be a decomposible ideal and let $a = \bigcap_{i=1}^{n} q_i$ be a minimal primary decomposition of a. Let $p_i = r(q_i), 1 \leq\leq i \leq n$. Then prove that the p_i are precisely the prime ideals which occur in the set of ideals $r(a : x), x \in A$ and hence independent of the particular decomposition of a.
    (ii) If the zero ideal of A is decomposible, then prove that the set D of zero divisors of A is the union of the prime ideals belonging to zero ideal.

21. (i) Let $A \subseteq B$ be integral domains, B is integral over A. Then prove that B is a field if and only if A is a field.
    (ii) State and prove going-down theorem.

(2 × 5 = 10 Weightage)

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