21U601 (Pages: 2) Name: ....

Reg.No: ....

SIXTH SEMESTER B.Sc. DEGREE EXAMINATION, APRIL 2024

(CBCSS - UG)

(Regular/Supplementary/Improvement)

CC19U MTS6 B10 / CC20U MTS6 B10 - REAL ANALYSIS

(Mathematics - Core Course)

(2019 Admission onwards)

Time : 2.5 Hours Maximum : 80 Marks

Credit : 5

Part A (Short answer questions)

Answer ***all*** questions. Each question carries 2 marks.

1. Give an example to show that a continuous function does not necessarily have an absolute maximum even though it has an absolute minimum on a set.
2. Let f be continuous on the interval [0, 1] to R and such that $f(0) = f(1)$. Prove that there exists a point c in I such that $f(c) = f(c + \frac{1}{2})$.
3. State preservation of interval theorem.
4. Prove that every Lipschitz function is uniformly continuous.
5. If $f(x) = x^2$ for $x \in [0, 5]$, calculate the Riemann sum of f corresponding to the partition $P = (0, 1, 3, 5)$ and tags are selected as the right end points of the subintervals.
6. If $f \in \mathfrak{R}[a, b]$, then prove that the value of the integral is uniquely determined.
7. State substitution theorem and hence evaluate $\int_0^2 t^2\sqrt{1 + t^3}\, dt$.
8. Prove that the set of all rational numbers is a null set.
9. Find the limit of the sequence of functions (f_n), where $f_n(x) = x^n$.
10. Define uniform norm. Find the uniform norm of the function f on [1, 3] where $f(x) = x^2$.
11. State Cauchy criterion for uniform convergence of sequence of functions.
12. Evaluate the improper integral $\int_1^\infty \frac{\ln x}{x^2}\, dx$.
13. Examine the convergence of $\int_1^5 \frac{dx}{\sqrt{x^4 - 1}}$.
14. Prove that $\beta(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$.

15. Express $\int_0^1 x^m(1-x^2)^n dx$ in terms of Beta function, where $m > 1$ and $n > -1$.

(Ceiling: 25 Marks)

Part B (Paragraph questions)

Answer ***all*** questions. Each question carries 5 marks.

16. Discuss the continuity of Thomae function.

17. Let I be a closed and bounded interval and let $f: I \to \mathrm{R}$ be continuous on I. If $\epsilon > 0$, then prove that there exists a continuous piecewise linear function $g_\epsilon: I \to \mathrm{R}$ such that $|f(x) - g_\epsilon(x)| < \epsilon$ for all $x \in I$.

18. Let f and g are in $\mathfrak{R}[a, b]$ and $f(x) \leq g(x)$, for all $x \in [a, b]$. Then prove that $\int_a^b f \leq \int_a^b g$.

19. Prove that if $f: [a, b] \to \mathrm{R}$ is a continuous function, then $f \in \mathfrak{R}[a, b]$.

20. Let (f_n) be a sequence of continuous functions on $A \subseteq \mathrm{R}$ and suppose that (f_n) converges uniformly to a function $f: A \to \mathrm{R}$. Then prove that f is continuous on A.

21. State and prove Weierstras M-test. Hence show that the series $\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} + \cdots$ converges uniformly.

22. Evaluate $\int_0^\infty e^{-x^2} dx$.

23. Prove the duplication formula: $\sqrt{\pi}\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m + \frac{1}{2})$.

(Ceiling: 35 Marks)

Part C (Essay questions)

Answer any ***two*** questions. Each question carries 10 marks.

24. Let f be a continuous function on a closed and bounded interval I. Prove that (i) f is bounded on I (ii) f has an absolute maximum on I.

25. State and prove additivity theorem.

26. Check whether $\int_\pi^\infty \frac{\sin x}{x} dx$ is conditionally convergent.

27. Show that even though the improper integral $\int_{-1}^5 \frac{dx}{(x-1)^3}$ does not converge, its Cauchy principal value exists.

(2 × 10 = 20 Marks)

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