C 83651 (Pages : 3) Name....................................

Reg. No. 29

SECOND SEMESTER M.Sc. DEGREE (CUCSS) EXAMINATION, JUNE 2015

Statistics

ST 2C 08—PROBABILITY THEORY

(2013 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *the questions.*
Weight 1 *for each question.*

1. State Radon-Nikodym theorem and mention its applications.
2. Prove that a sub-classes of independent classes are independent.
3. Prove that set of discontinuity points of a distribution function is countable.
4. Prove that pairwise independence of events does not imply mutual independence.
5. Give sequence $\{X_n\}$ of random variables that converges in distribution but does not converge in probability.
6. Let X_n be a sequence of independent random variables with $P\left[X_n = e^n\right] = 1/n^2$, $P[X_n = 0] = 1 - n^{-2}, n = 1, 2, \ldots\ldots$ Examine if the sequence converges almost surely to zero.
7. Examine the convergence in r^{th} mean for the sequence of random variables $\{X_n\}$ with $P\left[X_n = n^c\right] = 1/n,\ P[X_n = 0] = 1 - 2/n$ and $P\left[X_n = -n^c\right] = 1/n, n = 1, 2, \ldots\ldots$
8. Give an example of a sequence of events $\{A_n\}$ defined on a probability space such that $\sum_{n=1}^{\infty} P(A_n) = \infty$ but $P\left(\lim_{x \to \infty} A_n\right) = 0.$
9. Define characteristic function of a random variable. Prove that it is uniformly continuous.
10. Let $X_1, X_2, \ldots, X_n$ are mutually independent random variables. Then prove that the characteristic function of their sum is the product of the characteristic functions of the individual terms.

Turn over

11. State Lindeberg-Feller central limit theorem (CLT) and deduce Liapunov's CLT from this.

12. If $\{X_n\}$ and $\{Y_n\}$ are Martingales with respect to $\{F_n\}$, show that $X_n - Y_n$ is a Martingale w respect to $\{F_n\}$.

(12 × 1 = 12 weight

Part B

Answer any **eight** *questions.*
Weight 2 for each question.

13. Define tail σ-field and tail function. Give an example of tail function. State and prove Kolmogor zero-one law.

14. Define probability and deduce classical definition of probability.

15. Let X be a random variable defined on (Ω, F, P), prove that $P(X \le x)$ is non-decreasing r continuous function.

16. If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, prove that $X_n Y_n \xrightarrow{p} XY$ as $n \to \infty$.

17. Given a sequence $\{X_n\}$ of random variables with $P(X_n = 0) = 1 - 1/n^r$ and $P(X_n = n) = 1/n^r$, s that $X_n \not\to 0$ in r^{th} mean, but $X_n \to 0$ a.s.

18. Prove that if EX exists then weak law of large numbers holds for i.i.d. sequences.

19. Let $\{X_n\}$ be a sequence of independent r.v. with $P[X_k = k] = \frac{1}{2} k^{-\lambda} = P[X_k = -k]$ and $P[X_k = 0] = 1 - k^{-\lambda}$ $\lambda \ge 0$. Examine central limit theorem for every $\lambda \ge 0$.

20. Let $\{X_n\}$ be a sequence of independent random variables with following density. Examine i converges almost surely to 0 :

$$f_n(x) = \frac{n}{(1+ny)^2} \text{ for } y > 0 \text{; zero elsewhere.}$$

21. State and prove inversion theorem of characteristic function.

22. Let $\{Y_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $P(Y_n = +1) = \frac{1}{2} = P(Y_n = -1)$. D $X_n = \sum_{i=1}^{n} Y_i, X_0 = 0$. Show that $\{X_n, n \ge 1\}$ is a Martingale.

23. Define a sub-Martingale. If $\{Z_n, n \geq 1\}$ is a non-negative sub-Martingale, then prove that $P(\max\{Z_1, Z_2, ... Z_n\} > a) \leq E[Z_n]/a$ for $a > 0$.

24. State and prove Martingale convergence theorem. Mention one of its application.

(8 × 2 = 16 weightage)

Part C

Answer any **two** *questions.*
Weight 4 for each question.

25. (a) Prove that minimal σ-fields over independent classes, closed under finite intersection are independent.

(b) Let $\{A_n\}$ be a sequence of independent events. Show that $P(\overline{\lim}\, A_n) \rightarrow 1$ if $\sum_{n=1}^{\infty} P(A_n) = \infty$.

26. (a) State and prove Chebyshev's weak law of large numbers (WLLN). Determine whether WLLN holds for the following sequence of independent random variables.

$$P\left(X_n = \frac{n}{\log n}\right) = \frac{\log n}{2n} = P\left(X_n = -\frac{n}{\log n}\right) \text{ and } P(X_n = 0) = 1 - \frac{\log n}{n} \text{ for } n = 2, 3, ...$$

(b) If $X_n \xrightarrow{p} X$, prove that $X_n \xrightarrow{L} X$ as $n \rightarrow \infty$. When the converse holds ?

27. Let $\{X_n, Y_n\}$ $n = 1, 2, 3, ...$ be a sequence of pairs of random variables and let c be a constant. Then prove that $X_n \xrightarrow{L} X$, $Y_n \xrightarrow{p} c$, imply $X_n + Y_n \xrightarrow{L} X + c$ as $n \rightarrow \infty$.

28. State and prove Levy's continuity theorem of the characteristic function.

(2 × 4 = 8 weightage)