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Name...................

Reg. No.................

SECOND SEMESTER M.Sc DEGREE EXAMINATION, JULY 2016

(CUCSS-PG)

(Mathematics)

CC 15P MT2 C07 – REAL ANALYSIS II

(2015 Admission)

Time: Three Hours

Maximum : 36 Weightage

Part A

Short answer questions (1 - 14). Answer all questions.

Each question has one weightage.

1. Let A, B ∈ L(R^n, R^m). Prove that $\|A + B\| \leq \|A\| + \|B\|$.
2. Prove that the set of all invertible linear operators Ω on R^n is an open subset of L(R^n).
3. Define contraction mapping on a metric space and give an example of it.
4. State inverse function theorem.
5. Find the outer measure of the set of irrational numbers in the interval [-4, 4].
6. Is the set of natural numbers N measurable? Justify your answer.
7. Let A and B be measurable sets such that A ⊆ B. Prove that m*(A) ≤ m*(B).
8. Prove that constant functions are measurable.
9. Give an example where strict inequality occurs in Fatou's lemma.
10. Prove that if m*(A)= 0, then m*(A ∪ B) = m*(B).
11. Let f and g be bounded measurable functions defined on a set E of finite measure. If f ≤ g a.e., then show that $\int_E f \leq \int_E g$.
12. Let f be a monotonic increasing function on [a, b]. Prove that f is of bounded variation on [a, b].
13. Show that $D^+ [-f(x)] = - D_+ [f(x)]$.
14. Define absolute continuity. Give an example of an absolute continuous function.

(14 ×1 = 14 weightage)

Part B

Answer any seven from the following ten questions (15 – 24).

Each question has weightage two.

15. Let X be a vector space of dimension n. Prove that a set E of n vectors in X spans X if and only if E is independent.

16. Prove that every Borel set is measurable.

17. If f is measurable and $f = g$ a.e., then prove that g is measurable.

18. Prove that the outer measure is translation invariant.

19. Let f be a non-negative measurable function. Show that $\int f = 0$ implies $f = 0$ a.e..

20. Let f and g be integrable over E. Then prove that f + g is integrable over E and $\int_E (f+g) = \int_E f + \int_E g$.

21. If E_1 and E_2 are measurable, then prove that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

22. State and prove Monotone Convergence Theorem.

23. Let f be a function defined by $f(x) = \begin{cases} 0, & if\ x = 0 \\ x\sin\left(\frac{1}{x}\right), & if\ x \neq 0 \end{cases}$.

    Is f differentiable at x = 0? Justify your answer.

24. If f is absolutely continuous on [a, b], then prove that f is of bounded variation on [a, b].

(7× 2 = 14 weightage)

Part C

Answer any two from the following four questions (25 – 28).

Each question has weightage four.

25. Let $E \subseteq R^n$ be an open set and let $f : E \to R^m$ be a mapping differentiable at a point $x \in E$. Prove that the partial derivatives $(D_j f_i)(x)$ exist and

$$f'(x)\, e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i \quad \text{where } 1 \leq j \leq n.$$

26. (a) Prove that the outer measure of an interval is its length.

    (b) Let $\{E_i\}$ be a sequence of measurable sets. Prove that $m(\bigcup_i E_i) \leq \sum_i m(E_i)$.

27. (a) State and prove bounded convergence theorem.

    (b) State and prove Fatou's lemma.

28. Let f be an increasing real valued function on [a, b]. Prove that f is differentiable almost everywhere, the derivative f ' is measurable and $\int_a^b f'(x) \leq f(b) - f(a)$.

(2 × 4 = 8 weightage)