14

71328 (Pages : 2) Name....................................

Reg. No..................................

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2014

(CUCSS)

Mathematics

MT 3C 12—FUNCTIONAL ANALYSIS I

me : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Show that the metric space l^2 is not separable.
2. Show that every convergent sequence in a metric space is Cauchy.
3. Define n^{th} Dirichlet Kernel D_n and show that $\int_{-\Pi}^{\Pi} |D_n(t)|\, dt \to \infty$ as $n \to \infty$.
4. Show that the norm function on a normed linear space is continuous.
5. Show that the closed unit ball in l^2 is convex.
6. Let X be a normed space and $P \in BL(X)$ satisfy $P^2 = P$. Show that $\|P\| = 0$ *or* $\|P\| \geq 1$.
7. State Gram-Schmidt orthonormalization theorem.
8. Let $\{U_\alpha\}$ be an orthonormal set in an inner product space X and $x \in X$. Show that :

   $E_x = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$ ia a countable set.
9. Let X be an inner product space. Let $E \subset X$ and $x \in \bar{E}$. Show that there exists a best approximation from E to x iff $x \in E$.
10. Let X be a complex normed linear space and $A : X \to X$ be a linear map. Show that for all $x, y \in X$,

$$4\langle A(x), y\rangle = \langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle$$
$$+ i\langle A(x+iy), x+iy\rangle - i\langle A(x-iy), x-iy\rangle.$$

11. State Hahn-Banach separation theorem.
12. With usual notations, show that $C_0(T)$ is a Banach subspace of $C(T)$.
13. Show that every finite dimensional normed space is separable.
14. State uniform boundedness principle. (14 × 1 = 14 weightage)

Turn over

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. Show that a non-empty subset of a separable metric space is separable in the induced metric.

16. Show that the set of all simple measurable functions on a measurable subset E of R is dense in $L^{\alpha}(E)$.

17. Let X be a normed space. Show that if the subset $\{x \in X : \| x \| \leq 1\}$ of X is compact then X is finite dimensional.

18. Show that a linear functional f on a normed space X is continuous iff Z (f) is closed in X.

19. Let X and Y be inner product spaces. Show that a linear map $F : X \to Y$ satisfies $\langle F(x), F(y)\rangle = \langle x, y\rangle$ for all $x, y \in X$ iff it satisfies $\|f(x)\| = \|x\|$ for all $x \in X$, where the norms on X and Y are induced by the respective inner products.

20. State and prove Bessel's inequality.

21. Let X be an inner product space over K. Let $0 \neq x_1 \in X$ and $c_1 \in K$. Show that the element $x \in X$ of minimal norm satisfying $\langle x, x_1\rangle = c_1$ is $c_1 x_1 / \langle x_1, x_1\rangle$.

22. Let X be a normed space over K, Y be a subspace of X and $g \in Y'$. Show that there is some $f \in X'$ such that $f/_{y} = g$ and $\| f \| = \| g \|$.

23. Show that a normed space can be embedded as a dense subspace of a Banach space.

24. Let X be a normed space and E be a subset of X. Show that E is bounded in X iff f(E) is bounded in K for every $f \in X'$.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. Let E be a measurable subset of R. Show that, for $1 \leq p \leq \alpha$, the metric space $L^{p}(E)$ is complete.

26. Let Y be a closed subspace of a normed space X. For $x + Y$ in the quotient space X/Y, let :

    $\||x + Y\|| = \inf \{\|x + y\| : y \in Y\}$.

    Show that $\||\ \|\|$ is a norm on X/Y further show that a sequence $(x_n + Y)$ converges to $x + Y$ in X/Y iff there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X.

27. Show that a non-zero Hilbert space H is separable iff H has a countable orthonormal basis.

28. Let $X = \{x \in C([-\pi, \pi]) : x(\pi) = x(-\pi)\}$ with the sup norm. Show that the Fourier series of every x in a dense subset of X diverges at 0.

(2 × 4 = 8 weightage)