15P302 (Pages:2) Name..................

Reg. No..............

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2016

(CUCSS - PG)

(Mathematics)

CC15P MT3 C12 - FUNCTIONAL ANALYSIS I

(2015 Admission)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** Questions

Each question carries 1 weightage

1. Show that every convergent sequence in a metric space is Cauchy.
2. Let X and Y be normed spaces and $F: X \to Y$ be a linear map, then prove that if F is bounded on $\bar{U}(0,r)$ for some r, then $\|F(x)\| \leq \alpha\|x\|$, $\forall x \epsilon X$ and some $\alpha > 0$.
3. Define the n^{th} Dirichlet Kernel D_n and evaluate $\int_{-\pi}^{\pi} D_n(t)dt$.
4. Show that the norm function on a normed linear space is continuous.
5. Let $1 \leq p < r \leq \infty$. Prove that l^r is not contained in l^p.
6. Give an example of a discontinuous linear map from a normed space in to a normed space .
7. State and prove Schwarz inequality.
8. State Gram-Schmidt orthonormalization theorem.
9. Show that among all the l^p-spaces, $1 \leq p \leq \infty$ only l^2 is an inner product space .
10. Give an example of an uncountable orthonormal basis for a Hilbert space.
11. Let X be an inner product space. Show that if $E \subset X$ is convex then there exist at most one best approximation from E to any $x \in X$.
12. Let X be a normed space over K. Let $\{a_1, a_2, \ldots .. a_m\}$ be a linearly independent set in X. Show that there are $f_1, f_2, \ldots \ldots \ldots f_m$ in X' such that $f_j(a_i) = \delta_{ij}$, $1 \leq i, j \leq m$.
13. Show that a Banach space cannot have a denumerable basis.
14. State uniform boundedness principle.

(14×1=14 weightage)

Part B

Answer **any 7** Questions .

Each question carries 2 weightage

15. Show that set of all polynomials in one variable is dense in $C([a, b])$ with the sup metric.
16. Let $x \in L'[-\pi, \pi]$. Show that $\hat{x}$(n)→0 as $n \to \mp\infty$ where $\hat{x}(n)$ denotes the n^{th} Fourier coefficient of x.

17. Let X be a normed space. Then show that the following conditions are equivalent.

(a). Every closed and bounded subset of X is compact.

(b). The subset $\{ x \in X : \|x\| \leq 1\}$ of X is compact.

(c). X is finite dimensional.

18. Show that linear functional f on a normed space X is continuous iff $Z(f)$ is closed in X.

19. State and prove Bessel's inequality.

20. Let X be an inner product space, $\{u_1, u_2, \dots \dots\}$ be a countable orthonormal set in X and $k_1, k_2 \dots\dots \in K$. If X is a Hilbert space and $\sum_n |k_n|^2 < \infty$, then prove that $\sum_n k_n u_n$ converges in X.

21. Let X=C([-1,1]), $x(t) = 1 - t^2$, $x_0(t) = 1$, $x_1(t) = \cos \pi t$ for $t \in [-1,1]$. Show that the best approximation to x from span $\{x_0, x_1\}$ is $\frac{2}{3} + \frac{4x_1}{\pi^2}$.

22. Let $X = K^2$ with the norm $\| \quad \|_\infty$. Consider $Y = \{(x(1), x(2) \in X) : x(1) = x(2)\}$, and define $g \in Y$ by $g(x(1), x(2)) = x(1)$. Show that Hahn –Banach extensions of g to X are given by : $f\big(x(1), x(2)\big) = t\,x(1) + (1-t)x(2)$, where $t \in [0,1]$ is fixed .

23. Show that a normed space X is a Banach space iff every absolutely summable series of elements in X is summable in X .

24. Let X be a normed space. Then show that for every subspace Y of X and every $g \in Y'$, there is a Unique Hahn –Banach extension of g to X if and only if X' is strictly convex.

(7 × 2=14 weightage)

Part C

Answer **any 2** Questions .

Each question carries 4 weightage

25. Let E be a measurable subset of R . Show that for $1 \leq p \leq \infty$, the metric space $L^p(E)$ is complete.

26. Let H be a nonzero Hilbert space over K . Then prove that following conditions are equivalent.

(i) H has a countable orthonormal basis.

(ii) H is linearly isometric to K^n for some n, or to l^2.

(iii) H is separable.

27. State and prove Hahn – Banach separation theorem.

28. Let $X = \{x \in C([-\pi, \pi]) : x(\pi) = x(-\pi)\}$ with the sup norm. Show that the Fourier series of every X in a dense subset of X diverges at 0.

(2 × 4=8 weightage)

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