82514 (Pages : 2) Name..................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CUCSS)

Mathematics

MT 4E 02—ALGEBRAIC NUMBER THEORY

me : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*

Each question carries 1 *weightage.*

1. Express the polynomial $t_1^3 + t_2^3 + t_3^3$ in terms of elementary symmetric polynomials $(n = 3)$.
2. Let G be a free abelian group with basis x, y, z and let H be the subgroup of G generalised by $41x + 32y - 999z$, $16y + 3z$, $2y + 111z$. Find the order of G/H.
3. Express $Q(\sqrt{2}, \sqrt[3]{6})$ in the form $Q(\theta)$.
4. Prove that an algebraic integer is a national number if and only if it is a rational integer.
5. Let $\{\alpha_1, \ldots, \alpha_n\}$ be any Q-basis of K. Then prove that $\Delta[\alpha_1, \ldots, \alpha_n] = \det(T(\alpha_i \alpha_j))$.
6. Find a Z-basis for the integers of $Q(\sqrt[3]{5})$.
7. Prove that factorization into irreducibles is not unique in the ring of integers of $Q(\sqrt{15})$.
8. Prove : A prime in an integral domain D is always irreducible.
9. Prove that every Euclidean domain is a principal ideal domain.
10. Prove that every maximal ideal is a prime ideal.
11. If $\mathfrak{a} = \langle a \rangle$ is a principal non-zero ideal. Prove that $N(\mathfrak{a}) = |N(\alpha)|$.
12. Let L be an m-dimensional lattice in $\mathbb{R}^n$. Prove that $\mathbb{R}^n / L$ is isomorphic to $T^m \times \mathbb{R}^{n-m}$.
13. Let $K = Q(\theta)$ where $\theta^3 = 4$. Describe the map σ in this case.
14. Let $K = Q(\sqrt{5})$. Factorize the principal ideal $\langle 3 \rangle$ in the ring of integers of K.

(14 × 1 = 14 weightage)

Turn over

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. Let G be a free abelian group of rank r and H a subgroup of G. Then prove that G/H is finite i only if the ranks of G and H are equal.

16. Prove the algebraic integers form a subring of the field of algebraic numbers.

17. Prove that every number field passesses an integral basis.

18. Let $d \not\equiv 1 \pmod 4$ be a square free integer. Then prove that $Q(\sqrt{d})$ has an integral basis $\{1$ and discriminant 4d.

19. Let d be a square free negative integer with $d \neq -1, -3$. Let U be the group of units of the r integers of $Q(\sqrt{d})$. Show that $U = \{\pm 1\}$.

20. Prove that factorization into irreducibles is possible in a noetherian domain.

21. Let $x \in \infty$, the ring of integers in a number field K. Prove that x is a unit if and only if $N(x$

22. Let ∞ be the ring of integers in a number field. Prove that factorization of ∞ into irreduci unique if and only if ∞ is a principal ideal domain.

23. With usual notation prove : If $\alpha_1, \ldots\ldots, \alpha_n$ is a basis for k, over Q then $\sigma(\alpha_1), \ldots\ldots, \sigma(\alpha$ linearly independent over $\mathbb{R}$.

24. Compute the class number of $Q(\sqrt{-6})$.

(7 × 2 = 14 weig

Part C

Answer any **two** *questions.*
Each question carries 4 weightage.

25. Let $\zeta = e^{2\pi_i / p}$, p an odd prime. Prove that the ring of integers of $Q(\zeta)$ is $Z[\zeta]$.

26. Let $d < -1$ 1 be a square free integer. Prove that the ring of integers of $Q(\sqrt{d})$ is not Euc

27. If $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero ideals of the ring of integers of a number field, prove that $N(\mathfrak{a}\,\mathfrak{b}) = = N(\mathfrak{a}) \cdot N(\mathfrak{b})$.

28. Prove that the equation $x^4 + y^4 = z^2$ has no integer solutions with $x, y, z \neq 0$.

(2 × 4 = 8 wei