C 82512 (Pages : 3) Name..................................

Reg. No..................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2015

(CUCSS)

Mathematics

MT 4C 16—DIFFERENTIAL GEOMETRY

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*

Each question carries 1 weightage.

1. Show that the graph of any function $f:\mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F:\mathbb{R}^{n+1} \to \mathbb{R}$.

2. Find and sketch the gradient field of the function $f(x_1,x_2) = x_1 - x_2$.

3. Let $f:U \to \mathbb{R}$ be a smooth function and let $\alpha : I \to U$ be an integral curve of ∇f. Show that :

$$\left(\frac{d}{dt}\right)(f \circ d)(t) = \|\nabla f(\alpha(t))\|^2 \text{ for all } t \in I.$$

4. Sketch the cylinder over the graph of $f(x) = \cos x$.

5. Show that the two orientations on the unit n-sphere $x_1^2 + \ldots + x_{n+1}^2 = 1$ are given by :

$N_1(p) = (p,p)$ and $N_2(p) = (-p,p)$.

6. Prove that geodesics have constant speed.

7. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve and let X and Y be vector fields tangent to S along α. Verify that $(X + Y)' = X' + Y'$.

8. Compute $\nabla_v(f)$ where $f:\mathbb{R}^{n+1} \to \mathbb{R}, p \in \mathbb{R}^{n+1}, v \in \mathbb{R}_p^{n+1}$ where $f(x_1,x_2,x_3) = x_1,x_2,x_3^2$ and

$v = (1,1,1,a,b,c)\ (n = 2)$.

9. Find a global parametrization of the plane curve $x_1^2 + \frac{x_2^2}{4} = 1$. (You may choose the orientation).

Turn over

10. Find the length of the parametrized curve $\alpha : [0, 2\pi] \to \mathbb{R}^4$ given by $\alpha(t) = (\cos t, \sin t, \cos t, \sin$

11. Let $S \subset \mathbb{R}^{n+1}$ be an oriented n-surface, let $p \in S$. Define the second fundamental form of S

12. Let U be an open set in $\mathbb{R}^n$, let $\varphi : U \to \mathbb{R}^m$ be a smooth map, let $d\varphi$ be the differential c

Prove that the restriction $d\varphi_p$ of $d\varphi$ to $\mathbb{R}^n_p$ is a linear map $d\phi_p : \mathbb{R}^n_p \to \mathbb{R}^m_{\varphi(p)}$.

13. Let $Q : U_1 \to U_2$ and $\psi : U_2 \to \mathbb{R}^k$ be smooth where $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$. Verify the c

rule $d(\psi \circ \varphi) = d\psi \circ d\varphi$.

14. Let S be an n-surface in $\mathbb{R}^{n+k}$ $(k \geq 1)$. Let $p \in S$. Define the tangent space S_p at p.

(14 × 1 = 14 weigh

Part B

Answer any **seven** *questions.*

Each question carries 2 weightage.

15. Find the integral curve through $p = (a, b)$ of the vector field X on $\mathbb{R}^2$ given by $X(p) = (p, X$

where $X(x_1, x_2) = (x_2, x_1)$.

16. Sketch the tangent space at a typical point of the level set $f^{-1}(1)$ where $f(x_1, x_2, x_3) = x_1^2 + x_2^2$

17. Show that the set S of all unit vectors at all points of $\mathbb{R}^2$ forms a 3-surface in $\mathbb{R}^4$.

18. Show that the spherical image of an n-surface with orientation N is the reflection througl origin of the spherical image of the same n-surface with orientation $-$N.

19. Prove that, in an n-phase, parallel transport is path independent.

20. Let S be the unit n-sphere $\sum_{i=1}^{n+1} x_i^2 = 1$ oriented by the outward unit normal vector field. Prove

the Weingarten map of S is multiplication by -1.

21. Let $\alpha(t) = (x(t), y(t))$ $(t \in I)$ be a local parametrization of the plane curve C. Show that :

$$r \circ \alpha = \left(x' y'' - y' x''\right) / \left(x'^2 / y'^2\right)^{3/2}.$$

22. Let S be the ellipsoid $\left(x_1^2/a^2\right)+\left(x_2^2/b^2\right)+\left(x_3^2/c^2\right)=1$. Find the Gaussian curvature of S. $(abc \neq 0)$.

23. Show that the Weingarten map at each point of a parametrized n-surface is self-adjoint.

24. State and prove the Inverse Function Theorem for n-surfaces.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 *weightage.*

25. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : \mathrm{U} \to \mathbb{R}$ be smooth. Let $p \in \mathrm{U}$ be a regular point of f and let $c = f(p)$. Then prove : the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $\left[\nabla f(p)\right]^{\perp}$.

26. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $p \in \mathrm{S}$ and let $v \in \mathrm{S}_p$. Then prove there exists an open interval I containing O and a geodesic $\alpha : \mathrm{I} \to \mathrm{S}$ such that :

    (i) $\alpha(0) = p$ and $\dot{\alpha}(0) = \mathrm{U}$.

    (ii) If $\beta : \tilde{\mathrm{I}} \to \mathrm{S}$ is any other geodesic in S with $\beta(0) = p$ and $\dot{\beta}(0) = v$, then $\tilde{\mathrm{I}} \subset \mathrm{I}$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{\mathrm{I}}$.

27. Let η be the 1-form on $\mathbb{R}^2 - \{0\}$ defined by :

$$\eta = \frac{x_2}{x_1^2 + x_2^2}\,dx_1 + \frac{x_1}{x_1^2 + x_2^2}\,dx_2 .$$

Then prove that for $\alpha : [a,b] \to \mathbb{R}^2 - \{0\}$, any piecewise smooth closed parametrized curve in $\mathbb{R}^2 - \{0\}$ $\int_2 \eta = 2\pi k$ for some integer K.

28. Let $\varphi : \mathrm{U} \to \mathbb{R}^{n+1}$ be a parametrized n-surface in $\mathbb{R}^{n+1}$ and let $p \in \mathrm{U}$. Then show that there exists an open set $\mathrm{U}_1 \subset \mathrm{U}$ about p such that $\varphi(\mathrm{U}_1)$ is an n-surface in $\mathbb{R}^{n+1}$.

(2 × 4 = 8 weightage)