15P401 (Pages: 2) Name....................

Reg. No....................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2017

(CUCSS - PG)

(Mathematics)

CC15P MT4 C15 – FUNCTIONAL ANALYSIS II

(2015 Admission)

Time: Three Hours Maximum: 36 Weightage

Part I (Answer *all* questions – 14 x 1 = 14 weightage)

1. Let X be a normed space over K. Consider subsets U and V of X such that $U \subset V + kU, k \in K$. Prove that for every $x \in U$, there is a sequence (v_n) in V such that $x - (v_1 + kv_2 + \cdots + k^{n-1}v_n) \in k^n U,\ n = 1,2,\ldots$

2. Let X be a normed space and $f : X \to K$ be linear. Prove that f is a closed map if and only if f is continuous.

3. Find the relation between $\sigma(A)$ and $\sigma(A^{-1})$

4. Let X_0 be a dense subspace of a normed space X. For $x' \in X'$, let $F(x')$ denote the restriction of x' to X_0. Show that F is a linear isometry from X' onto X_0'.

5. Define a reflexive normed space. Is l^1 reflexive? Why?

6. Let X be a normed space. $z \in X$ and $f \in X'$. Show that $T: X \to X$ defined by $T(x) = f(x)z$ is a compact linear map.

7. Let X be an inner product space. $E \subseteq X$. Show that $E^{\perp}$ is a closed subspace of X.

8. Let $(x_n) \in H$. Prove that $x_n \to x$ if and only if $x_n \xrightarrow{w} x$ and $\limsup ||x_n|| \leq ||x||$

9. Define adjoint of $A \in BL(H)$ and show that $(A + B)^* = A^* + B^*$.

10. Let $A \in BL(H)$. Prove that A is Normal if and only if $||A(x)|| = ||A^*(x)||, \forall x \in H$

11. Show that each orthogonal projection on a Hilbert space is a positive operator.

12. If x_1 and x_2 are eigen vectors of a normal operator corresponding to distinct eigen values show that they are orthogonal.

13. Let A be a self adjoint compact operator on a non-zero Hilbert space, show that $||A|| \text{ or } -||A||$ is an eigen value of A.

14. Give an example of Hilbert –Schmidt operator on $H = l^2$

Part II (Answer any *seven* questions – 7 x 2 =14 weightage)

15. Let X be a normed space. Prove that the projection P on X is a closed map if and only if R(P) and Z(P) are closed in X.

16. State and prove two-norm theorem

17. Let X be a normed space $A \in BL(X)$ be of finite rank. Show that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$

18. Show that the map $T: BL(X, Y) \to BL(Y', X')$ mapping F to F′ is linear and norm preserving, $F \in BL(X, Y)$

19. Let X be a reflexive normed space. Prove that X′ is reflexive.

20. Let X be a uniformly convex normed space and (x_n) be a sequence in X such that $||x_n|| \to 1$ and $||x_n + x_m|| \to 2$ as $n, m \to \infty$. Show that (x_n) is Cauchy.

21. Prove that every Hilbert space is reflexive.

22. Let H be a Hilbert space. $A \in BL(H)$. Show that $R(A) = H$ if and only if A^* is bounded below.

23. State and prove generalized Schwartz Inequality.

24. Let $A \in BL(H)$. Show that $\sigma_e(A) \subseteq \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \{k: \bar{k} \in \sigma_e(A^*)\}$

Part III (Answer any *two* questions – 2 x 4 = 8 weightage)

25. State and prove Open mapping theorem. Give example to show that it does not hold if X is not a Banach space.

26 Prove that the dual of K^n with the norm $\|.\|_p$ is linearly isometric to K^n with the norm $\|.\|_q$, where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$

27. State and prove unique Hahn Banach extension theorem.

28. Let H be a Hilbert space and $A \in BL(H)$ be a Hilbert –Schmidt operator. Show that A and A^* are compact.

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