16P401 (Pages: 2) Name: …………………

Reg.No: …………………

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2018

(Regular/Supplementary/Improvement)

(CUCSS - PG)

CC15P MT4 C15-FUNCTIONAL ANALYSIS II

(Mathematics)

(2015 Admission onwards)

Time: 3 Hours Maximum: 36 Weightage

Part A

Answer ***all*** questions. Each question carries **1** weightage.

1. Let X be a normed space and $f: X \to K$ be linear. Prove that f is closed iff f is continuous.
2. Prove or disprove: "Comparable norms preserves completeness".
3. Let X be a normed space and$A \in BL(X)$beinvertible. Showthat $\sigma(A^{-1}) = \{ k^{-1}: k \in \sigma(A)\}$.
4. Let X be a Banach space, $A \in BL(X)$ and $\|A^p\| < 1$ for some positive integer p. Prove that zero is not a spectral value of the operator $I - A$.
5. Let X and Y be two normed spaces, $k \in K$and $A, B \in BL(X, Y)$. Prove that $(A + B)' = A' + B'$ and $(kA)' = k A'$
6. Prove or disprove: "Dual of a separable normed space need not be separable".
7. Prove that the normed spacel^p is reflexive for $1 < p < \infty$.
8. Let H be a Hilbert space and $f, g \in H'$. Define $\langle f, g\rangle' = \langle T(g), T(f)\rangle$ where $T: H' \to H$ defined by $T(f) =$ the representer of f in H. Prove that $\langle \ , \ \rangle'$ is an inner product on H' and $\langle f, f\rangle' = \|f\|^2, \forall f \in H'$.
9. Let H be a Hilbert space over $\mathbb{C}$ and $A \in BL(H)$. Prove that there exist unique self adjoint operators B and C on H such that $A = B + iC$.
10. Let H be a Hilbert space and $A \in BL(H)$. Prove that$\|A^*A\| = \|A\|^2 = \|AA^*\|$.
11. Prove or disprove: "Every normal operatoris unitary".
12. Prove that the composition of two positive operators need not be positive.
13. Give an example of a self adjoint operator whose eigen spectrum is empty.
14. Let H be a self-adjoint compact operator on a non zero Hilbert space and $A \in BL(H)$. Prove that $\|A\|$ or $-\|A\|$ is an eigen value of A.

(14 x 1 =14 weightage)

Part B

Answer any ***seven*** questions. Each question carries **2** weightage.

15. Give an example of a closed linear map which is not continuous. Does this contradict the closed graph theorem?

16. Let $X = l^p$, $1 \le p \le \infty$ and define $A(x) = \left(x(1), \frac{x(2)}{2}, \frac{x(3)}{3}, \ldots \right)$ for $x \in l^p$. Find $\sigma\,(A)$.

17. Let X be a Banach space. Prove that the set of all invertible operators is open in $BL(X)$ and the map $A \to A^{-1}$ is continuous on this set.

18. Let X be a normed space and $\{x_1', x_2', \ldots, x_m'\}$ be a linearly independent subset of X'. Prove that there exist $x_1, x_2, \ldots, x_m \in X$ such that $x_j'(x_i) = \delta_{ij}$,for $i, j = 1, 2, \ldots, m$, where $\delta_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \ne j \end{cases}$.

19. State and prove projection theorem.

20. Prove that every bounded sequence in a Hilbert space has a weak convergent subsequence.

21. Let H be a Hilbert space and $A \in BL(H)$. Prove that there exist a unique $B \in BL(H)$ such that $\langle A(x), y \rangle = \langle x, B(y) \rangle$, for all $x, y \in H$.

22. Let X be an inner product space and $A, B \in BL(X)$ with A self-adjoint. Prove that $AB = 0$ iff $R(A) \perp R(B)$.

23. Let H be a Hilbert space and $A \in BL(H)$. Prove that $A = 0$ iff $\langle A(x), x \rangle = 0$ for all $x \in H$.

24. Prove that the set of all compact operators on a Hilbert space H is closed in $BL(H)$.

(7 x 2 =14 weightage)

Part C

Answer any ***two*** questions. Each question carries **4** weightage.

25. State and prove open mapping theorem. Also prove that the completeness property cannot be dropped from the assumption.

26. Let X be a normed space and $A \in BL(X)$ be of finite rank. Prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.

27. State and prove Riesz Representation theorem for continuous linear functionals on a Hilbert space.

28. Let H be a Hilbert space over $\mathbb{C}$ and A be a continuous linear map on H. Prove that

    (i) $k \in \sigma\,(A)$if and only if $\bar{k} \in \sigma\,(A^*)$.

    (ii) $\sigma_e(A) = \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \{ k \in \mathbb{C} : \bar{k} \in \sigma_e(A^*) \}$.

(2 x 4 =8 weightage)

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