16P403 (Pages :2) Name …………………..

Reg.No ………………..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2018

(Regular/Supplementary/Improvement)

(CUCSS – PG)

CC15 PMT4 E02-ALGEBRAIC NUMBER THEORY

(Mathematics)

(2015 Admission onwards)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer ***all*** questions. Each question carries ***1*** weightage.

1. Express the polynomial $t_1^3 + t_2^3 + t_3^3$ in terms of elementary symmetric polynomials(n =3).
2. Find the order of the group G/H where G is free abelian with Z-basis x, y, z and H is generated by x + 3y – 5z, 2x – 4y, 7x + 2y – 9z.
3. Find all monomorphisms $Q(\sqrt[3]{7}) \to C$.
4. Find θ such that $Q(\theta) = Q(\sqrt{2}, i)$.
5. Let K = Q(ξ) where $\xi = e^{2\pi i/5}$. Calculate $N_K(\alpha)$ and $T_K(\alpha)$ for $\alpha = \xi + \xi^2$.
6. Is 10 = (3 + i)(3 – i) = 2.5 an example of non-unique factorization in $\mathbb{Z}[i]$? Justify your answer.
7. Show that a prime in an integral domain D is always irreducible.
8. Let $x \in \mathfrak{O}$. If N(x) is a rational prime, prove that x is irreducible in $\mathfrak{O}$.
9. Let K be the number field Q(ξ) where $\xi = e^{2\pi i/p}$ for an odd prime p. If I is the ideal generated by λ = 1-ξ in the ring of integers $\mathbb{Z}[\xi]$ of K, then show that $I^{p-1} = \langle p \rangle$ and N(I) = p.
10. Prove that the ring of integers of a number field is Noetherian.
11. Prove that, in $\mathbb{Z}[\sqrt{-17}]$, the element 2 is irreducible but not prime.
12. Sketch the lattice in $\mathbb{R}^2$ generated by (1, 1) and (1, 0) and a fundamental domain for the lattice.
13. Show that if L is an n-dimensional lattice in R^n then R^n/L is isomorphic to the n-dimensional torus.
14. For a given number field k, define the $\mathbb{R}$-algebra L^{st} and the map $\sigma : k \to L^{st}$.

(14 x 1 = 14 weightage)

Part B

Answer any ***seven*** questions. Each question carries ***2*** weightage

15. Let R be a ring. Show that every symmetric polynomial in $R[t_1,t_2,---,t_n]$ is expressible as a polynomial with coefficients in R in the elementary symmetric polynomials S_1, S_2 --- , S_n.

16. Show that every subgroup H of a free abelian group G of rank n is free of rank $s \leq n$.

17. Show that an algebraic number α is an algebraic integer iff its minimum polynomial over Q has coefficients in Z.

18. Let d be a square free rational integer. Show that the integers of $Q(\sqrt{d})$ are :

    (i) $Z[\sqrt{d}]$ if $d \not\equiv 1 \bmod(4)$.

    (ii) $Z\left[\frac{1}{2} + \frac{1}{2}\sqrt{d}\right]$ if $d \equiv 1 \bmod(4)$.

19. Show that an integral domain D is noetherianiff D satisfies the ascending chain condition.

20. Show that factorization into irreducible is not unique in the ring of integers of $Q(\sqrt{15})$.

21. Show that the non-zero fractional ideals of D, where D is the ring of integers of a number field K of degree n, form an abelian group under multiplication.

22. Show that an additive group of R^n is a lattice iff it is discrete.

23. State and prove Minkowski's theorem.

24. Let L be an n-dimensional lattice in R^n with basis $\{e_1, e_2, \text{---}, e_n\}$, show that if $e_i = (a_{1i}, a_{2i}, \text{---}, a_{ni})$, then the volume of the fundamental domain T of L defined by the basis is $\upsilon(T) = |\det a_{ij}|$.

(7 x 2 = 14 weightage)

Part C

Answer any ***two*** questions. Each question carries ***4*** weightage

25. Prove that the ring of integers of $\mathbb{Q}(\xi)$ is $\mathbb{Z}[\xi]$ if $\xi = e^{2\pi i/p}$, where p is an odd prime.

26. Let **D** be a domain in which factorization into irreducible is possible. Prove that factorization into irreducible is unique iff every irreducible is prime.

27. Prove that the field $Q(\sqrt{d})$ in norm-Euclidean for d = -1, -2, -3, -7 and -11.

28. a). With usual notations show that if $\alpha_1, \alpha_2, \text{---}, \alpha_n$ is a basis for K over Q then $\sigma(\alpha_1), \sigma(\alpha_2), \text{---}, \sigma(\alpha_n)$ are linearly independent over R.

    b). Let D be the ring of integers of a number field K of degree n. Show that if $\mathbf{a} \neq 0$is an ideal of D then **a** contains an integer α with $|N(\alpha)| \leq \left(\frac{2}{\pi}\right)^t N(\mathbf{a}) \sqrt{|\Delta|}$, where Δ is the discriminant of K.

(2 x 4 = 8 weightage)

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