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Reg. No..................................

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, FEBRUARY 2013

(CUCSS)

Mathematics

MAT 1C 03—REAL ANALYSIS—I

(2010 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A (Short Answer Questions) (1 —14)

Answer **all** *questions.*
Each question has 1 weightage.

1. Construct a bounded set of real numbers with exactly one limit point.

2. For $x, y \in \mathbb{R}^1$, let $d(x, y) = |x^2 - y^2|$. Is d a metric on $\mathbb{R}$? Justify your answer.

3. Let E be a non-empty set of real numbers which is bounded above and let $y = \sup E$. Prove that $y \in \overline{E}$.

4. Is a finite set closed ? Justify your answer.

5. Prove that the limit of a function is unique.

6. Construct a function which has a simple discontinuity at every rational point.

7. Let f be a differentiable function on $[a, b]$. Prove that f is continuous on $[a. b]$.

8. Let f be a continuous function and $f \geq 0$ on $[a, b]$. If $\int_a^b f\, dx = 0$, then prove that $f(x) = 0$ for all $x \in [a, b]$.

9. Let f_1, f_2 be bounded functions and α be a monotonic increasing function on $[a, b]$. Prove that if f_1, f_2 are Reimann-Steiltjes integrable with respect to α on $[a, b]$, then $f_1 + f_2$ is Reimann-Steiltjes integrable with respect to α on $[a, b]$.

10. Let f be a bounded function and α be a monotonic increasing function on $[a, b]$. If the partition P′ is a refinement of the partition P of [a, b], then prove that.

$$L(P, f, \alpha) \leq L(P', f, \alpha).$$

11. Let γ be defined on $[0, 2\pi]$ by $\gamma(t) = e^{it}$. Prove that γ is rectifiable.

Turn over

12. Give an example of a convergent series of continuous functions with a discontinuous limit.

13. Prove that uniformly convergent sequence of bonded functions is uniformly bounded.

14. Define equicontinuous family of functions and give an example of it.

(14 × 1 = 14 weigh

Part B

Answer any **seven** *from the following ten questions* (15 –24).
Each question has weightage 2.

15. Prove that the set of all integers is countable.

16. Prove that compact subsets of a metric space are closed.

17. Let f be a continuous real valued function on a metric space X. Prove that th $Z(f)=\{x \in X : f(x)=0\}$ is a closed subset of X.

18. Let $[x]$ denote the largest integer less than or equal to x. What type of discontinuities do function $[x]$ have ?

19. If f is a real valued differentiable function on (a, b). If $f'(x) \geq 0$ for all $x \in (a,b)$, then prov f is monotonic increasing on (a, b).

20. Let f be a bounded function and α be a monotonic increasing function on $[a, b]$. Prove that Reimann-Steiltjes integrable with respect to α on $[a, b]$, then $|f|$ is Reimann-Steiltjes integ with respect to α on $[a, b]$ and $\left|\int_a^b f\, d\alpha\right| \leq \int_a^b |f|\, d\alpha.$

21. For $1 < s < \infty$, define $\zeta(s)=\sum_{n=1}^{\infty} \frac{1}{n^s}$. Prove that $\zeta(s)=\frac{s}{s-1}-s\int_1^{\infty}\frac{x-[x]}{x^{s+1}}\,dx,$ where $[x]$ d the greatest integer less than or equal to x.

22. For what values of x does the series $\frac{1}{1+n^2 x}$ converge absolutely.

23. Prove that the series $\sum_{n=1}^{\infty}(1-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval.

24. Let K be compact, $f_n \in C(K)$ $n = 1, 2, 3, \ldots$ and let $\{f_n\}$ be pointwise bounded and equicontinuous on K. Prove that $\{f_n\}$ is uniformly bounded on K.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *from the following four questions* (25 –28).
Each question has weightage 4.

25. (a) Prove that countable union of countable sets is countable.

(b) Prove that the cantor set is perfect.

26. (a) Prove that a mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

(b) Prove that continuous image of a connected space is connected.

27. (a) State Taylor's theorem.

(b) Let f be a bounded function, α be monotonic increasing function and α' is Reimann integrable on $[a, b]$. Prove that f is Reimann-Steiltjes integrable with respect to α on $[a, b]$ if and only if $f\,\alpha'$ is Reimann integrable on $[a, b]$.

28. Let γ be a curve on $[a, b]$ and let γ' be continuous on $[a, b]$. Prove that γ is rectifiable and

$$\wedge(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

(2 × 4 = 8 weightage)