20P402 (Pages : 2) Name : ....

Reg. No : ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2022

(CBCSS-PG)

CC19P MTH4 C15 - ADVANCED FUNCTIONAL ANALYSIS

(Mathematics - Core Course)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part - A

Answer ***all*** questions. Each question carries 1 weighatge.

1. State true or false and justify. " The eigen vectors corresponding to distinct eigenvalues of a bounded operator are linearly independent. "
2. Let X be a Banach space and $A \in L(X)$ be invertible. Then prove that $\sigma(A^{-1}) = \{\frac{1}{\lambda} : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of the operator A.
3. Let H be a Hilbert space and A be a bounded operator on H such that $\langle A(x), x\rangle \in \mathbb{R}$ for every $x \in H$. Prove that A is a symmetric operator.
4. If $A \geq 0$, then prove that $P(A) \geq 0$, for every polynomial $P(\lambda)$ with non-negative coefficients.
5. Let P be an orthoprojection on a Hilbert space H. Then prove that $0 \leq P \leq I$, where I is the identity operator on H.
6. State true or false and justify. " A linear map is a continuous map if and only if it is a closed map ".
7. State Banach-Steinhaus theorem.
8. Define Banach algebra and give an example.

(8 × 1 = 8 Weightage)

Part - B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

Unit I

9. Let T be a compact operator on a Banach space X. If $\lambda \neq 0$ and $\Delta_\lambda = X$, then prove that $\ker(T_\lambda) = 0$, where $\Delta_\lambda = \text{Im}(T - \lambda I)$ and $T_\lambda = T - \lambda I$.
10. Consider the operator $T : l_2 \to l_2$ defined by $T(\mathbf{x}) = (0, x_1, x_2, ...)$, $\mathbf{x} = (x_1, x_2, ...) \in l_2$. Find the spectrum of T.

11. Let H be a Hilbert space and A be a symmetric operator on H. Prove that $\|A\| = \sup_{x\neq 0} \frac{|\langle A(x),x\rangle|}{\|x\|^2}$.

Unit II

12. Let H be a Hilbert space and $\{A, A_0, A_1, A_2, ...\}$ be a collection of bounded operators on H. If $A_0 \leq A_1 \leq A_2 \leq ... \leq A_n \leq ... \leq A$, then prove that there exists a bounded operator B such that $A_n(x) \to B(x)$ for all $x \in H$.

13. Let H be a Hilbert space and A be a bounded operator on H such that $mI \leq A \leq MI$. Let $\phi, \psi \in \mathcal{K}[m, M]$ such that $\psi(t) \leq \phi(t)$ for all $t \in [m, M]$. If $\{Q_n\}$ and $\{P_n\}$ be sequences of polynomials such that $Q_n(t) \searrow \psi(t)$ and $P_n(t) \searrow \phi(t)$ for all $t \in [m, M]$, then prove that $\lim_{n\to\infty} Q_n(A) \leq \lim_{n\to\infty} P_n(A)$. Here $\mathcal{K}[m, M]$ is the set of all piecewise continuous bounded functions which are monotone decreasing limits of continuous functions.

14. Let H be a Hilbert space and a, b, m, M be real numbers such that $a < m \leq M < b$. If A is a bounded operator on H such that $mI \leq A \leq MI$, then construct the spectral integral of A.

Unit III

15. State and prove the Banach open mapping theorem.

16. Let X and Y be Banach spaces and $A : X \to Y$ be a closed graph operator. If $\text{Dom}(A) = X$, then prove that A is a bounded operator.

17. Let X be a Banach space, K be a closed convex set in X and $x_0 \notin K$. Prove that there is a $f \in X^*$ such that $f(x_0) > \sup_{x\in K} f(x)$.

(6 × 2 = 12 Weightage)

Part-C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Let T be a compact operator on a Banach space X. If $\lambda \neq 0$, then prove that $\Delta_\lambda = \text{Im}(T - \lambda I)$ is a closed subspace of X.

19. State and prove the first Hilbert-Schmidt theorem.

20. Let H be a Hilbert space and A be a bounded operator on H such that $A \geq 0$. Prove that there exists a unique operator $B \geq 0$ on H such that $B^2 = A$.

21. Let $\mathcal{A}$ be a Banach algebra. Prove that there exists an equivalent norm $|\cdot|$ on $\mathcal{A}$ such that $|x \cdot y| \leq |x|.|y|$, for all $x, y \in \mathcal{A}$.

(2 × 5 = 10 Weightage)

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