21P401 (Pages: 2) Name: ....

Reg.No: ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2023

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 E08 - COMMUTATIVE ALGEBRA

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer any ***all*** questions from each unit. Each question carries 1 weightage.

1. Define ideal and quotient ring.
2. Prove that every non-units of A is contained in a maximal ideal.
3. Define annihilator of an A-module M and faithful module.
4. Let q be a p-primary ideal of Ring A and $x \in A$. If $x \notin p$, then prove that $(q : x) = q$.
5. Define primary decomposition and minimal primary decomposition.
6. Prove that Z (the set of integers) is integrally closed in Q (the set of rational numbers).
7. If A is Noetherian and $\phi : A \rightarrow B$ is a onto homomorphism, then prove that B is Noetherian.
8. Prove that if A is an Artin ring, then its nilradical is equal to the Jacobson radical.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. Let $f : A \rightarrow B$ be a ring homomorphism and C is the set of all contracted ideals in A. Then prove that $C = \{a : a^{ec} = a\}$.
10. Prove that there is a natural isomorphism $HOM(A, M) \cong M$, for any A-module M.
11. Suppose N is finitely generated as a B-module and that B is finitely generated as an A-module. Then prove that N is finitely generated as a A-module.

UNIT - II

12. Let $g : A \rightarrow B$ be a ring homomorphism such that $g(s)$ is a unit in B, for all $s \in S$. Then prove that there exists a unique ring homomorphism $h : S^{-1}A \rightarrow B$ such that $g = h \circ f$, where $f : A \rightarrow S^{-1}A$ defined by $f(x) = x/1$.
13. Prove that the $S^{-1}A$-modules $S^{-1}(M/N)$ and $S^{-1}M/S^{-1}N$ are isomorphic.

14. Let M be an A-module. If $M_m = S^{-1}M = 0$, for all maximal ideal m of A, then prove that $M = 0$ (Here $S = A - m$).

UNIT - III

15. State and prove going-up theorem.

16. Let $0 \to M' \overset{\alpha}{\to} M \overset{\beta}{\to} M'' \to 0$ be an exact sequence of A-modules. Then prove that M is Noetherian if and only if M', M'' are Noetherian.

17. Let A be Noetherian ring and M be a finitely generated A-module. Then prove that M is Noetherian.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. (i) Prove that the nilradical of A is the intersection of all prime ideals of A.
    (ii) Prove that x belongs to Jacobson Radical of A if and only if $1 - xy$ is a unit in A for all $y \in A$.

19. Let $M' \overset{u}{\to} M \overset{v}{\to} M'' \to 0$ be a sequence of A-modules and homomorphisms. Then prove that the sequence $M' \overset{u}{\to} M \overset{v}{\to} M'' \to 0$ is exact if and only if for all A-modules N, the sequence $0 \to HOM(M'', N) \overset{\overline{v}}{\to} HOM(M, N) \overset{\overline{u}}{\to} HOM(M', N)$ is exact, where $\overline{v}(f) = f \circ v, \overline{u}(g) = g \circ u$.

20. (i) Prove that $S^{-1}(r(a)) = r\left(S^{-1}(a)\right)$, for any ideal a in A.
    (ii) If M is a finitely generated A-module and S is a multiplicatively closed subset of A, then prove that $S^{-1}(Ann(M)) = Ann\left(S^{-1}M\right)$.

21. (i) Prove that an Artin ring has only a finite number of maximal ideals.
    (ii) Prove that a ring A is Artin if and only if A is Noetherian and $dimA = 0$.

(2 × 5 = 10 Weightage)

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