21P404 (Pages: 2) Name: ………………………..

Reg. No……………………..

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2023

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 E09 – DIFFERENTIAL GEOMETRY

(Mathematics)

(2019 Admission onwards)

Time: Three Hours Maximum: 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Define the level set and graph of any function $f: \mathbb{R}^{n+1} \to \mathbb{R}$
2. Find and sketch the gradient field of the function $f(x_1, x_2) = x_1 - x_2^2$
3. Define Gauss map, Spherical image. Find the spherical image of the cone $-x_1^2 + x_2^2 = 0$ oriented by $\frac{\nabla f}{\|\nabla f\|}$.
4. For each $a, b, c, d \in \mathbb{R}$, prove that the parametrized curve $\alpha(t) = (\cos(at+b), \sin(at+b), ct+d)$ is geodesic in the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$
5. Compute$\nabla_v f$ where $f: \mathbb{R}^2 \to \mathbb{R}, v \in \mathbb{R}_p^2$, $f(x_1, x_2) = x_1^2 - x_2^2, v = (1,1, \cos\theta, \sin\theta)$
6. Compute $\int_\alpha \sum_{i=1}^{n+1} x_i dx_i$ where $\alpha: [0,1] \to \mathbb{R}^{n+1}$ is such that $\alpha(0) = (0,0, \dots, 0)$ and $\alpha(1) = (1,1, \dots, 1)$
7. Find a global parametrization of the curve $ax_1 + bx_2 = c$, $(a, b) \neq 0$ oriented by $\frac{\nabla f}{\|\nabla f\|}$
8. Define parametrized n −surface in $\mathbb{R}^{n+k}, k \geq 0$

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions form each unit. Each question carries 2 weightage.

Unit-I

9. Find the integral curve through $p = (1,1)$ of the vector field $\boldsymbol{X}$ on $\mathbb{R}^2$ given by $\boldsymbol{X}(p) = (p, X(p))$ Where $X((x_1, x_2)) = (x_2, x_1)$.
10. Show that the maximum and minimum values of the function $g(x_1, x_2 \dots x_n) = \sum_{i,j=1}^{n+1} a_{ij} x_i x_j$ on the unit sphere $x_1^2 + x_2^2 + \cdots x_n^2 = 1$ where $(a_{ij})$is a symmetric $n \times n$ matrix of real numbers, are the eigen values of the matrix (a_{ij})
11. Let $S = f^{-1}(c)$ be an $n-$ surface in $\mathbb{R}^{n+1}$, where $f: U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0 \; \forall q \in S$ and let $\boldsymbol{X}$ be a smooth vector field on U whose restriction to S is a tangent

vector field on S.If $\alpha: I \to U$ is any integral curve of $\boldsymbol{X}$ such that $\alpha(t_0) \in S$ for some $t_0 \in S$.Then prove that $\alpha(t) \in S \;\; \forall t \in I$

Unit-II

12. Compute the Weingarten map for the circular cylinder $x_2^2 + x_3^2 = a^2$ in $\mathbb{R}^3, a \neq 0$, $N = \frac{\nabla f}{\|\nabla f\|}$.
13. Let $\alpha(t) = \big(x(t), y(t)\big), t \in I$ be a local parametrization of the oriented plane curve C. Show that the curvature $\kappa \circ \alpha = \frac{x'y''-x''y'}{[(x')^2+(y')^2]^{3/2}}$
14. Prove that $\eta = \frac{-x_2}{x_1^2+x_2^2}dx_1 + \frac{x_1}{x_1^2+x_2^2}dx_2$ is not an exact 1-form on $\mathbb{R}^2\backslash\{0\}$.

Unit-III

15. Let V be a finite dimensional real vector space with dot product and $L: V \to V$ be a self adjoint linear operator. Prove that there exist an orthonormal basis for V consisting of eigen vectors of L.
16. Find the Gaussian curvature of the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$ $(a, b, c$ all $\neq 0)$ oriented by its outward normal
17. Prove that on each compact oriented $n -$ surface S in $\mathbb{R}^{n+1}$ there exist a point p such that the second fundamental form at p is definite.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Let S be a compact connected oriented $n -$ surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f: \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$.Then prove that the Gauss map maps S onto the unit sphere S^n.
19. a) Let S be an $n -$ surface in $\mathbb{R}^{n+1}$ and $\alpha: I \to S$ be a parametrized curve in S, let $t_0 \in I, \boldsymbol{v} \in S_{\alpha(t_0)}$. Then prove that there exist a unique vector field $\mathbb{V}$, tangent to S along α, which is parallel and has $\mathbb{V}(t_0) = \boldsymbol{v}$.

    b) Let S be an $n -$ surface in $\mathbb{R}^{n+1}$, let $p, q, \in S$ and let α be a piece wise smooth parametrized curve from p to q. Then prove that the parallel transport $P_\alpha: S_p \to S_q$ along α is a vector space isomorphism which preserves dot products.
20. Let C be a plane curve. Then prove that C has a global parametrization if and only if C is connected.
21. State and prove inverse function theorem for n-surfaces.

(2 × 5 = 10 Weightage)

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