23P101 (Pages: 2) Name: ....

Reg.No: ....

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2023

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH1 C01 - ALGEBRA - I

(Mathematics)

(2019 Admission onwards)

Time: 3 Hours Maximum : 30 Weightage

Part A

Answer *all* questions. Each carries 1 weightage.

1. Describe all symmetries of a line segment in $\mathbb{R}$.
2. State Burnside's Formula. How many distinguishable necklaces can be made using 7 different colored beads of same size.
3. Find the order of (3,10,9) in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$.
4. Prove that no group of order 30 is simple.
5. For a prime p, prove that every group G of order p^2 is abelian.
6. Prove that every group G' is a homomorphic image of a free group G.
7. Find all zeros of $(x^3 + 2x + 5)(3x^2 + 2x)$ in $\mathbb{Z}_7$.
8. Let F be a ring of all functions mapping $\mathbb{R}$ to $\mathbb{R}$ and let C be the subring of F consisting of all constant functions in F. Is C an ideal in F? Justify your answer.

($8 \times 1 = 8$ Weightage)

Part B

Answer any *two* questions from each unit. Each carries 2 weightage.

Unit 1

9. If m is a square free integer, that is not divisible by the square of any prime, then prove that every abelian group of order m is cyclic.
10. Let X be a G- set and let $x \in X$. Then prove that $|Gx| = (G : G_x)$. If $|G|$ is finite, then prove that $|Gx|$ is a divisor of G.
11. Define center of a group and commutator subroups of a group with examples. Prove that commutator subgroup is a normal subgroup of the group.

Unit 2

12. If G has a compostition series and N is a proper normal subgroup of G, then prove that there exists a composition series containing N.

13. State and prove third Sylow theorem.

14. If H and K are finite subgroups of a group G, then prove that $|HK| = \frac{(|H|)(|K|)}{|H \cap K|}$

Unit 3

15. Determine all groups of order 10 upto isomorphism using group presentations.

16. Prove that an element $a \in F$ is a zero of $f(x) \in F[x]$ if and only if $(x - a)$ is a factor of $f(x) \in F[x]$.

17. Is the polynomial $x^5 + x^4 + x^3 + x^2 + x + 1$ is irreducible. Prove that cyclotomic polynomials are irreducible.

($6 \times 2 = 12$ Weightage)

Part C

Answer any *two* questions. Each carries 5 weightage.

18. (a) Let H be a subgroup of G. Then prove that left coset multiplication is well defined by the equation $(aH)(bH) = (ab)H$, if and only if H is a normal subgroup of a group G.

    (b) Find the order of the group $\mathbb{Z}_{12} \times \mathbb{Z}_{18}/\langle(4, 3)\rangle$ and the order of the element $(3, 4) + \langle(4, 3)\rangle$ in the group.

19. (a) Let p be a prime, and let G be a finite group and p divides $|G|$. Then prove that G has an element of order p and consequently, a subgroup of order p.

    (b) Let G be a finite group. Then G is a p-group if and if only $|G|$ is a power of p.

20. (a) State class equation and find the class equation D_4.

    (b) Prove that no group of order 48 is simple.

21. (a) State and prove Eisentein criterion.

    (b) Show that $x^2 + 8x - 2$ is irreducible over $\mathbb{Q}$. Is $f(x)$ irreducible over $\mathbb{R}$? over $\mathbb{Z}$?.

($2 \times 5 = 10$ Weightage)

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