

## THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2015

(CUCSS)

Mathematics

MT 3C. 13—TOPOLOGY—II

Time : Three Hours

Maximum : 36 Weightage

## Part A

Answer all questions.  
Each question has weightage 1.

1. Prove that, if a product is non-empty, then each projection is onto.
2. Let  $\tau$  be the product topology on the set  $\prod_{i \in I} X_i$ , where  $\{(X_i, \tau_i) : i \in I\}$  is an indexed collection of topological spaces. Prove that the family of all subsets of the form  $\pi_i^{-1}(V_i)$  for  $V_i \in \tau_i, i \in I$  is a subbase for  $\tau$ .
3. Prove that a topological product is  $T_0$  if each coordinate space is  $T_0$ .
4. Let  $X = \prod_{i \in I} X_i$ , where each  $X_i$  is a topological space and let  $x \in X$ . Suppose  $\{x_n\}$  is a sequence in  $X$  with the property that for each  $i \in I$ , the sequence  $\{\pi_i(x_n)\}$  converges to  $\pi_i(x)$  in  $X_i$ . Prove that  $\{x_n\}$  converges to  $x$  in  $X$ .
5. Prove that the evaluation function of a family of functions is one-to-one if and only if that family distinguishes points.
6. Give an example of a second countable Hausdorff space which is not metrizable.
7. Prove that in a simply connected space, any two points having the same initial and final points are path homotopic.
8. Show that if  $p : E \rightarrow B$  is a covering map, then  $p$  is an open map.
9. Prove that the map  $p : S' \rightarrow S'$  given by  $\hat{p}(z) = z^2$  is a covering map.
10. Prove that a continuous real-valued function on a countably compact space is bounded and attains its extrema.
11. Prove that a first countable countably compact space is sequentially compact.

Turn over

12. Let  $X$  be Hausdorff and locally compact at a point  $x$  in  $X$ . Prove that the family of compact neighbourhoods of  $x$  is a local base at  $x$ .
13. Let  $A$  be a subset of a metric space  $(X, d)$  such that  $A$  is complete with respect to the metric induced on it. Prove that  $A$  is closed in  $X$ .
14. Prove that the set of irrationals is of second category in the real line  $\mathbb{R}$  with the usual topology.

(14 × 1 = 14 marks)

**Part B**

*Answer any seven questions.  
Each question has weightage 2.*

15. Prove that any continuous real-valued function on a closed subset of a normal space can be extended continuously to the whole space.
16. Let  $\{X_i : i \in I\}$  be an indexed family of sets and let  $X = \prod_{i \in I} X_i$ . Prove that a subset of  $X$  is a box and only if it is the intersection of a family of walls.
17. Let  $\mathcal{S}$  be a subbase for a topological space  $X$ . Prove that  $X$  is completely regular if and only if for each  $V \in \mathcal{S}$  and for each  $x \in V$ , there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$  and  $f(y) = 1$  for all  $y \notin V$ .
18. Let  $\{f_i : X \rightarrow Y_i \mid i \in I\}$  be a family of functions which distinguishes points from closed sets in  $X$ . Prove that the corresponding evaluation function  $e : X \rightarrow \prod_{i \in I} Y_i$  is open when regarded as a function from  $X$  onto  $e(X)$ .
19. Prove that path homotopy ( $\simeq_p$ ) is an equivalence relation on the set of paths in a topological space  $X$ .
20. Let  $P : E \rightarrow B$  be a covering map and let  $p(e_0) = b_0$ . Prove that any path  $f : [0, 1] \rightarrow B$  beginning at  $b_0$  has a lifting to a path  $\tilde{f}$  in  $E$  beginning at  $e_0$ .
21. Prove that a subset of Euclidean space is compact if and only if it is closed and bounded.
22. Among all Hausdorff compactifications of a Tychonoff space, prove that the Stone-Čech compactification is the largest compactification up to a topological equivalence.
23. Prove that every compact metric space is complete.
24. Prove that the function  $e : \hat{X} \times \hat{X} \rightarrow \mathbb{R}$  defined by  $e([\hat{x}], [\hat{y}]) = \lim_{n \rightarrow \infty} d(x_n, y_n)$  (where  $\hat{x} = \{x\}$  and  $\hat{y} = \{y_n\}$ ) is a metric on  $\hat{X}$ , where  $(X, d)$  is a metric space.

(7 × 2 = 14 weightage)

## Part C

Answer any **two** questions.  
Each question has weightage 4.

25. Prove that a product of spaces is connected if and only if each co-ordinate space is connected.
26. State and prove Urysohn's metrization theorem.
27. Prove that the fundamental group of the circle is infinite cyclic.
28. Prove that a topological space is compact if and only if there exists a closed sub-base  $\mathcal{C}$  for  $X$  such that every family of  $\mathcal{C}$  having the finite intersection property has a non-empty intersection.

(2 × 4 = 8 weightage)