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C 3545 (Pages : 4) Name.....................................

Reg. No.................................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2016

(CUCSS)

Mathematics

MT 4C 16—DIFFERENTIAL GEOMETRY

Time : Three Hours Maximum : 36 Weightage

Standard notation as in the prescribed text is followed.

Part A

Answer **all** *questions. Each question carries weightage* 1.

1. Sketch the level sets $f^{-1}(c)$ at the heights indicated

$$f(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2 \,;\, c = -1, 0.$$

2. Find and sketch the gradient field of the function $f(x_1, x_2) = \left(x_1^2 - x_2^2\right) / 4$.

3. Show by example that the set of vectors tangent at a point p of a level set might be all of $\mathbb{R}_p^{n+1}$.

4. Show that the set S of all unit vectors at all points of $\mathbb{R}^2$ forms a 3-surface in $\mathbb{R}^4$.

5. Show that if S is a connected n-surface in $\mathbb{R}^{n+1}$ and $g : S \to \mathbb{R}$ is continuous and takes on only finitely many values, then g is constant.

6. Describe the spherical image of the paraboloid $-x_1 + x_2^2 + x_3^2 = 0$ (Choose your orientation).

7. Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed, then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$.

8. Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve. Let X be a vector field tangent to S along α. Verify that

$$(f\,X)' = f'\,X + f\,X'$$

for all smooth functions f along α.

Turn over

9. Compute ∇f where $f:\mathbb{R}^2 \to \mathbb{R}$ and $v \in \mathbb{R}^2_p$, $p \in \mathbb{R}^2$.

$$f(x_1, x_2) = x_1^2 - x_2^2,\ v = (1, 1, \cos\theta, \sin\theta).$$

10. Let C be an oriented plane curve and $p \in C$ with $k(p) \neq 0$. Define the circle of curvature at p.

11. Find the length of the given parametrized curve $d:[0, 2\pi] \to \mathbb{R}^3$, where $\alpha(t) = (\sqrt{2}\cos 2t, \sin 2t, \sin 2t)$.

12. Let S be an oriented 2-surface in $\mathbb{R}^3$ and let $p \in S$. Show that for each $v, \omega \in S_p$

$$L_p(\vartheta) \times L_p(\omega) = k(p)\, \vartheta \times \omega.$$

13. Let $Q: U_1 \to U_2$ and $\psi: U_2 \to \mathbb{R}^k$ be smooth. Verify the chain rule $d(\psi \circ \phi) = d\psi \circ d\psi$.

14. Show that if $S = f^{-1}(c)$ is an n-surface in $\mathbb{R}^{n+k}$ and $p \in S$, then the tangent space S_p to S at p is equal to the kernal of df_p.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions. Each question carries weightage 2.*

15. Find the integral curve through $p(0,1)$ of the vector field X on $\mathbb{R}^2$ given b

$$X(p) = (p, X(p)) \text{ where } X(x_1, x_2) = \left(-2x_1, \frac{-1}{2}x_2\right).$$

16. Show that the maximum and minimum values of the function $g(x_1, \ldots\ldots, x_{n+1}) = \sum_{i,j=1}^{n+1} a_{ij}\, x_i$ on the unit n-sphere $\sum_{i=1}^{n+1} x_i^2 = 1$ where (a_{ij}) is a symmetric $n \times n$ matrix of real numbers, are t eigenvalues of the matrix (a_{ij}).

17. Let S be an n-surface in $\mathbb{R}^{n+1}$, let X be a smooth tangent vector field on S and let $p \in S$. Then prove the existence of the maximal integral curve of X through p.

18. Show that if the spherical image of a connected n-surface is a single point, then S is contained in an n-plane.

19. For $\theta \in \mathbb{R}$, let $\alpha_\theta : [0, \pi] \to S^2$ be the parametrized curve in the unit sphere S^2 from the north pole $p = (0, 0, 1)$ to the south pole $q = (0, 0, -1)$, defined by $\alpha_\theta(t) = (\cos\theta \sin t, \sin\theta, \sin t, \cos t)$. Let $v = (p, 1, 0, 0) \in S_p$. Then compute $P_{\alpha_0}(v)$.

20. Let S be an n-surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field N. Let $p \in S$ and $v \in S_p$. Let $\alpha : I \to S$ be a parametrized curve with $\dot\alpha(t_0) = v$ for some $t_0 \in I$. Then prove that $\ddot\alpha(t_0) . N(p) = L_p(v) . v$.

21. Let $g : I \to \mathbb{R}$ be a smooth function and let C denote the graph of g. Show that the curvature of C at the point $(t, g(t))$ is $g''(t)/(1 + g'_{(t)}\,2)^{3/2}$ for an appropriate choice of orientation.

22. Find the Gaussian curvature of the ellipsoid $x_1^2/4 + x_2^2/4 + x_3^3/9 = 1$.

23. Show that the Weingarten map at each point of a parametrized n-surface in $\mathbb{R}^{n+1}$ is self-adjoint.

24. State and prove inverse function theorem for n-surfaces.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries weightage 4.

25. Let S be a compact, connected oriented n-surface in $\mathbb{R}^{n+1}$. Prove that the Gauss map maps S onto the unit n-sphere S^n.

Turn over

26. Let C be a connected, oriented plane curve and let $\beta : I \to C$ be a unit speed global parametrizati
C. Then prove that β is either one-to-one or periodic. Further show that β is periodic iff C
compact.

27. Let S be a compact oriented n-surface in $\mathbb{R}^{n+1}$. Prove : There exists a point $p \in S$ such that t
second fundamental form at p is definite.

28. Let S be an n-surface in $\mathbb{R}^{n+1}$ and let $f : S \to \mathbb{R}^k$. Suppose that $f \circ g$ is smooth for each le
parametrization, $\varphi : U \to S$. Then prove that f is smooth.

(2 × 4 = 8 weightag