15P403

(Pages :2) Name ....

Reg. No.....................

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, MARCH 2017

(CUCSS – PG)

(Mathematics)

CC15P MT4 E02-ALGEBRAIC NUMBER THEORY

(2015 Admission)

Time : Three Hours Maximum : 36 Weightage

Part A
(Answer *all* Questions)
Each question carries 1 weightage

1. Express $t_1^3 + t_2^3$ in terms of elementary symmetric polynomials.
2. Find a $\mathbf{Z}$ basis for the integers of $\mathbf{Q}(\sqrt[3]{5})$.
3. Find a ring which is not noetherian.
4. Is the number $\frac{1+\sqrt{17}}{2\sqrt{-19}}$ an algebraic integer. Justify your answer.
5. Find integral basis and discriminant for the field $\mathbf{Q}(\sqrt{2}, i)$.
6. Let $\mathbf{K} = \mathbf{Q}(\xi)$, where $\xi = e^{2\pi i/p}$ for a rational p. In the ring of integers $\mathbf{Z}[\xi]$, show that $\alpha \in \mathbf{Z}[\xi]$ is a unit if and only if $N_K(\alpha) = \pm 1$.
7. Let $\{\alpha_1, \alpha_2, --- , \alpha_n\}$ be any $\mathbf{Q}$-basis of $\mathbf{K}$. Then prove that $\Delta[\alpha_1, \alpha_2, --- , \alpha_n] = \det(T(\alpha_i, \alpha_j))$.
8. Let $\mathbf{K} = \mathbf{Q}(\xi)$ where $\xi = e^{2\pi i/5}$. Calculate $N_K(\alpha)$ and $T_K(\alpha)$ for $\alpha = 1 + \xi + \xi^2 + \xi^3 + \xi^4$.
9. Let $\mathfrak{O}$ be the ring of integers of a number field and let $\mathfrak{p}$ be a non-zero prime ideal of $\mathfrak{O}$. Prove that $\mathfrak{p}$ is a maximal ideal.
10. Prove that every principal ideal domain is a unique factorization domain.
11. Calculate the class number of $\mathbf{Q}(\sqrt{d})$ for d square free and $-10 \le d \le 10$.
12. Sketch the lattice in $\mathbf{R}^2$ generated by (-1, 0) and (0, 1) and a fundamental domain for the lattice.
13. Give L^{st} and σ explicitly for $\mathbf{K} = \mathbf{Q}(\sqrt[4]{5})$.
14. Find principal ideals $\mathfrak{a}$, $\mathfrak{b}$ in $\mathbf{Z}[\sqrt{-6}]$ such that $\mathfrak{a}\langle 2, \sqrt{-6}\rangle = \mathfrak{b}\langle 3, \sqrt{-6}\rangle$.

Part B

(Answer *any seven* Questions)

Each question carries 2 weightage

15. Let **K** be a number field. Prove that the discriminant of any basis for **K** is rational and non-zero.
16. Prove that every number field **K** possesses an integral basis.
17. Let $\mathfrak{O}$ be the ring of integers of a number field **K**. Prove that the additive group of $\mathfrak{O}$ is a free abelian group of rank n, where n is the degree of **K**.
18. Prove that the discriminant of $\mathbf{Q}(\xi)$, where $\xi = e^{2\pi i/p}$ and p an odd prime is $(-1)^{(p-1)/2}.p^{p-2}$.
19. Prove that a complex number θ is algebraic iff the ring $\mathbf{Z}[\theta]$ is a finitely generated abelian group.
20. Compute the prime factorization of the ideal $\langle 18 \rangle$ in $\mathbf{Z}[\sqrt{-17}\,]$.
21. Prove that factorization of elements of $\mathfrak{O}$ into irreducible is unique iff every ideal of $\mathfrak{O}$ is principal.
22. Prove that every ideal $\mathfrak{a}$ of $\mathfrak{O}$ with $\mathfrak{a} \neq 0$ has a **Z**-basis $\{\alpha_1, \alpha_2, ---, \alpha_n\}$ where n is the degree of **K** . Also prove that $N(\mathfrak{a}) = \left|\frac{\Delta[\alpha_1, \alpha_2, ---, \alpha_n]}{\Delta}\right|^{1/2}$, where Δ is the discriminant of **K**.
23. If X is a bounded subset of $\mathbf{R}^n$ and $v(X)$ exists, and if $v(\upsilon(X)) \neq v(X)$, then prove that υ/X is not injective.
24. Let **K** be a number field. Prove that the only roots of unity in **K** are $\pm \xi^s$ for integers s.

Part C

(Answer *any two* Questions)

Each question carries 4 weightage

25. Let **K** be a number field. Then prove that $\mathbf{K} = \mathbf{Q}(\theta)$ for some algebraic number θ.
26. Let d < -11 be a square free integer. Prove that the ring of integers of $\mathbf{Q}(\sqrt{d}\,)$ is not Euclidean.
27. Let **D** be a domain in which factorization into irreducible is possible. Prove that factorization into irreducible is unique iff every irreducible is prime.
28. (a) Let L be an m-dimensional lattice in $\mathbf{R}^n$. Then prove that $\mathbf{R}^n/L$ is isomorphic to $\mathbf{T}^m \times \mathbf{R}^{n-m}$.

    (b) If **M** is a lattice in $\mathbf{L}^{st}$ of dimension s + 2t having fundamental domain of volume V, and if $c_1, c_2, ---, c_{s+t}$ are positive real numbers whose product $c_1 c_2 --- c_{s+t} > \left(\frac{4}{\pi}\right)^t V$. Prove that in M there exist a non-zero element $x = (x_1, x_2, ---, x_{s+t})$ such that $|x_1| < c_1, |x_2| < c_2, ---, |x_s| < c_s$; $|x_{s+1}|^2 < c_{s+1}, ---, |x_{s+t}|^2 < c_{s+t}$.

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