SECOND SEMESTER M.Sc. DEGREE EXAMINATION, JUNE 2014

(CUCSS)

Mathematics

MT 2C 07—REAL ANALYSIS—II

Time : Three Hours   Maximum : 36 Weightage

Part A

Short answer questions 1–14. Answer all questions.
Each question has 1 weightage.

1. Let X be a vector space and let dim $X = n$. Prove that a set E of n vectors spans X if and only if E is independent.

2. Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and let $x \in \mathbb{R}^n$. Prove that $A'(x) = A$.

3. Define contraction mapping on a metric space and give an example of it.

4. Let $f = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f_1(x,y) = e^x \cos y, \quad f_2(x,y) = e^x \sin y.$$

   Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$.

5. Find the Lebesgue outer measure of the set $\{1 \pm \dfrac{1}{2^n} : n = 1, 2, 3, \ldots\}$.

6. Let A and B be measurable sets such that $A \subseteq B$. Prove that $m^*(A) \leq m^*(B)$.

7. Is the set of irrational numbers in the interval $[1, 100]$ measurable? Justify your answer.

8. Prove that constant functions are measurable.

9. Give an example where strict inequality occur in Fatou's lemma.

10. Show that if f is integrable, then so is $|f|$.

11. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure. If $f_n \to f$ a.e., then prove that $\{f_n\}$ converges to f in measure.

12. Show that $D^+[-f(x)] = -D_+ f(x)$.

13. Show that if $a \leq c \leq b$, then $T_a^b = T_a^c + T_c^b$.

14. Prove that sum of two absolutely continuous functions is continuous. $(14 \times 1 = 14 \text{ weightage})$

Turn over

Part B

*Answer any **seven** from the following **ten** questions (15–24).*
Each question has weightage 2.

15. Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$. Prove that Ω is an open subset of $L(\mathbb{R}^n)$.

16. Let

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0) \\ \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0). \end{cases}$$

Prove that $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ exist at every point of $\mathbb{R}^2$.

17. If E_1 and E_2 are measurable, then prove that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

18. Prove that sum of two measurable functions defined on a same measurable set is measurable.

19. Prove that the characteristic function χ_E is measurable if and only if E is measurable.

20. Let $E_1, E_2, \ldots, E_n$ be disjoint measurable sets and let $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$. Prove that

$$\int \varphi = \sum_{i=1}^{n} a_i m(E_i).$$

21. Let E be a measurable set and let f, g be integrable over E. Prove that $f + g$ is integrable over E and

$$\int_E f + g = \int_E f + \int_E g.$$

22. Let f be a function defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin(\frac{1}{x}) & \text{if } x \neq 0. \end{cases}$$

Is f differentiable at $x = 0$? Justify your answer.

23. If f is of bounded variation on $[a, b]$, then prove that $f'(x)$ exists for almost all x in $[a, b]$.

24. If f is absolutely continuous on $[a, b]$, then prove that f is of bounded variation on $[a, b]$.

$(7 \times 2 = 14 \text{ weightage})$

Part C

Answer any two from the following four questions (25–28).
Each question has weightage 4.

25. (a) Let E be an open subset of $\mathbb{R}^n$ and **f** maps E into $\mathbb{R}^m$. If f is differentiable at a point $\mathbf{x} \in E$, then prove that the partial derivatives $(D_j f_i)(\mathbf{x})$ exist.

   (b) If $[A]$ and $[B]$ are n by n matrices, then prove that

$$\det([A][B]) = \det[A]\det[B].$$

26. (a) Prove that outer measure of an interval is its length.

   (b) Let $\{E_i\}$ be a sequence of measurable sets. Prove that

$$m\left(\bigcup_i E_i\right) \le \sum_i m(E_i).$$

27. (a) State and prove bounded convergence theorem.

   (b) Let $\{f_n\}$ be a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E. Prove that

$$\int_E f_n \le \underline{\lim} \int_E f_n.$$

28. Let f be an increasing real valued function on the interval $[a, b]$. Prove that f is differentiable almost everywhere, the derivative f' is measurable and

$$\int_a^b f'(x) \le f(b) - f(a).$$

(2 × 4 = 8 weightage)