# Queueing Theory

**Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics**

by

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# CERTIFICATE

This is to certify that the project entitled **"Queueing Theory"** submitted to Postgraduate and Research Department of Mathematics in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics, is a bonafide record of project work done by **Ms.ANITTA RAJESH(CCAWMMS002)** during the period of her study in the Postgraduate and Research Department of Mathematics, Christ College (Autonomous), Irinjalakuda, under my supervision and guidance during the year 2023-2024

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I hereby declare that the project work entitled "**Queueing Theory**" submitted to Christ College(Autonomous), Irinjalakuda in partial fulfilment of the requirement for the award of Master Degree of Science in Mathematics is a record of project work done by me during the period of my study in the Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda.

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First, there are no words to adequately acknowledge the wonderful grace that my Redeemer has given me. There are many individuals who have come together to make this project a reality. I greatly appreciate the inspiration; support and guidance of all those people who have been instrumental for making this project a success.

I express my deepest thanks to my guide Dr. Seena Varghese, Assistant Professor, Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda, for the invaluable guidance, patience and expertise throughout the course of the project. Her priceless and meticulous support has inspired me in innumerable ways.

I take this opportunity to express my thanks to our beloved principal Fr. Dr. Jolly Andrews CMI, who gave me the golden opportunity to do this wonderful project on the topic "**Queueing Theory**"

I mark my word of gratitude to Dr. Seena V, Head of the Department and all other teachers of the department for providing me the necessary facilities to complete this project on time.

Our HoD Seena. V, deserves a special word of thanks for her invaluable and generous help in preparing this project in $LAT_EX$.

I want to especially thank all the faculty of the library for providing various facilities for this project.

Words cannot express the love and support I have received from my parents, whose encouragement has buoyed me up from the beginning till the end of this work.

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# An Introduction to the p-adic Numbers

Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics

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# CERTIFICATE

This is to certify that the project entitled **"An Introduction to the p-adic Numbers"** submitted to Postgraduate and Research Department of Mathematics in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics, is a bonafide record of original project work done by **Ms.ANJANA V R (CCAWMMS003)** during the period of her study in the Postgraduate and Research Department of Mathematics, Christ College (Autonomous), Irinjalakuda, under my supervision and guidance during the year 2022-2024

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I hereby declare that the project work entitled "**An Introduction to the p-adic Numbers**" submitted to Christ College(Autonomous), Irinjalakuda in partial fulfilment of the requirement for the award of Master Degree of Science in Mathematics is a record of project work done by me during the period of my study in the Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda.

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# ACKNOWLEDGEMENT

First, there are no words to adequately acknowledge the wonderful grace that my Redeemer has given me. There are many individuals who have come together to make this project a reality. I greatly appreciate the inspiration; support and guidance of all those people who have been instrumental for making this project a success.

I express my deepest thanks to my guide Mrs Tintu mol Sunny, Assistant Professor, Department of Mathematics, Christ College(Autonomous), Irinjalakuda, who guided me faithfully through this entire project. I have learned so much from her, both in the subject and otherwise. Without her advice, support and guidance, it find difficult to complete this work.

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Our HoD Dr. Seena V, deserves a special word of thanks for her invaluable and generous help in preparing this project in $LAT_EX$.

I want to especially thank all the faculty of the library for providing various

facilities for this project.

Words cannot express the love and support I have received from my parents, whose encouragement has buoyed me up from the beginning till the end of this work.

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# ALGEBRAIC TOPOLOGY

**Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics**

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I hereby declare that the project work entitled " **ALGEBRAIC TOPOLOGY** " submitted to Christ College(Autonomous), Irinjalakuda in partial fulfilment of the requirement for the award of Master Degree of Science in Mathematics is a record of project work done by me during the period of my study in the Postgraduate and Research Department of Mathematics , Christ College(Autonomous), Irinjalakuda.

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I express my deepest thanks to my guide Dr. Seena V, Assistant Professor, Postgraduate and Research Department of Mathematics , Christ College(Autonomous), Irinjalakuda, who guided me faithfully through this entire project. I have learned so much from her, both in the subject and otherwise. Without her advice, support and guidance, it find difficult to complete this work.

I take this opportunity to express my thanks to our beloved principal Fr. Dr. Jolly Andrews CMI, who gave me the golden opportunity to do this wonderful project on the topic "**ALGEBRAIC TOPOLOGY**"

I mark my word of gratitude to all teachers of the department for providing me the necessary facilities to complete this project on time.

My project guide Dr. Seena V, deserves a special word of thanks for her invaluable and generous help in preparing this project in $LAT_EX$.

I want to especially thank all the faculty of the library for providing various facilities for this project.

Words cannot express the love and support I have received from my parents, whose encouragement has buoyed me up from the beginning till the end of this work.

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# Contents

# Introduction

In mathematics,*Topology* is from the Greek word '*topos*'means place and '*logos*' means study. *Topology* is concerned with the properties of geometric objects that are without preserved under continuous deformations such us stretching, twisting, crumpling and bending, but not tearing or gluing.

A topological space is a set endowed with a structure called a *topology*, which allows defining continues deformation of subspaces and more generally, all kinds of continuity. Eucledian space and more generally, metric spaces are examples of a topological space, as any distance or metric defines a topology. The deformations that are considered as topology are homeomorphisms and homotopies. A property that is invariant under such deformations is a topological property. Basic examples of topological properties are the dimension; compactness; connectedness.

Algebra is one of the broad parts of mathematics, together with number theory, geometry and analysis. In its most general form, algebra is the study of mathematical symbols and the rules for manipulating these symbols. It includes everything from the elementary equations solving to the study of abstraction such as group, rings and fields. The Greek mathematician Diophantus has traditionally been known as 'the father of algebra'. The more basic parts of algebra are called elementary algebra; the most abstract parts are called abstract algebra or modern algebra. Elementary algebra is generally considered to be essential for any study of mathematics, science or engineering as well as such equations on medicine and economics.

Abstract algebra is a major area in advance mathematics, study primarily by professional mathematician.

Algebraic topology is a branch of mathematics that uses tools from extract algebra to study topological spaces. The basic goal is to find the algebraic invariants that classify topological spaces upto homeomorphism, through usually most classify up to homotopy equivalence. Although algebraic topology primarily uses algebra to study topological problems, using topology to solve algebraic problems is sometimes also possible.

In this project we discuss about covering spaces, simplicial complexes and applications of algebraic topology.

# Outline of the Project

Apart from the introductory chapter, we have described our work in four chapters.

**Chapter 1** covers the basic definitions in topology and abstract algebra such that vector spaces, topological spaces, local homeomorphism etc.

In **Chapter 2**, covers the topic covering spaces, $G$-spaces, $G$- sets, properties of covering spaces.

In **Chapter 3**, deals with simplicial complexes and simplicial approximation .

In **Chapter 4**, covers the applications of algebraic topology in various areas.

# Representation Theory

Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics

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This is to certify that the project entitled **"Representation Theory"** submitted to Postgraduate and Research Department of Mathematics in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics, is a bonafide record of project work done by **Ms. ANJU .M.S (CCAWMMS005)** during the period of her study in the Postgraduate and Research Department of Mathematics, Christ College (Autonomous), Irinjalakuda, under my supervision and guidance during the year 2023-2024

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I express my deepest thanks to my guide Mr. Anand M S, Assistant Professor, Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda, who guided me faithfully through this entire project. I have learned so much from him, both in the subject and otherwise. Without his advice, support and guidance, it find difficult to complete this work.

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I want to especially thank all the faculty of the library for providing various

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# THE PAGERANK ALGORITHM

Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics

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This is to certify that the project entitled **"The PageRank Algorithm"** submitted to Postgraduate and Research Department of Mathematics in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics, is a bonafide record of project work done by **Ms.ANJU SIBI (CCAWMMS006)** during the period of her study in the Postgraduate and Research Department of Mathematics, Christ College (Autonomous), Irinjalakuda, under my supervision and guidance during the year 2023-2024.

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**Anju sibi**

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# PELL'S EQUATION

**Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics**

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# CERTIFICATE

This is to certify that the project entitled **"PELL'S EQUATION "** submitted to Postgraduate and Research Department of Mathematics in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics, is a bonafide record of project work done by **Ms.APARNA K M (CCAWMMS007)** during the period of her study in the Postgraduate and Research Department of Mathematics, Christ College (Autonomous), Irinjalakuda, under my supervision and guidance during the year 2023-2024

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I express my deepest thanks to my guide Dr.Seena Varghese, Assistant Professor, Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda, for the invaluable guiance, patience and expertise throughout the course of the project. Her priceless and meticulous support has inspired me in innumerable ways.

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facilities for this project.

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# A study on Fractional Programming

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# Matroid Theory

**Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics**

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I hereby declare that the project work entitled "**Matroid Theory**" submitted to Christ College(Autonomous), Irinjalakuda in partial fulfilment of the requirement for the award of Master Degree of Science in Mathematics is a record of project work done by me during the period of my study in the Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda.

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First, there are no words to adequately acknowledge the wonderful grace that my Redeemer has given me. There are many individuals who have come together to make this project a reality. I greatly appreciate the inspiration; support and guidance of all those people who have been instrumental for making this project a success.

I express my deepest thanks to my guide Dr. Seena V, Head and Assistant Professor, Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda, who guided me faithfully through this entire project. I have learned so much from her, both in the subject and otherwise. Without her advice, support and guidance, it find difficult to complete this work.

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I want to especially thank all the faculty of the library for providing various

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**Athulya P R**

# Contents

# Introduction

Matroid theory is a branch of mathematics that abstracts and generalizes the notion of linear independence from vector spaces to more general sets. It is a rich field of study with connections to geometry, topology, combinatorics, and optimization. The fundamental concept in matroid theory is the matroid itself, which is a combinatorial structure that captures the essence of independence in a manner that is applicable to a wide variety of mathematical contexts.

The concepts of a Matroid theory was introduced in 1935 by Hassler Whitney, a mathematician who aimed to generalize the notion of independence beyond the confines of vector spaces. The foundation of Matroid theory resets on a set coupled with a collection of subsets, defined by specific axioms that encapsulate the essence of independence. These axioms mirror the properties of independence in vector spaces: the most basic being that subsets of independence sets are independent, and that one can extend an independent set to a larger independent set if it is not maximal.

Central to Matroid theory is the concept of a basis, which is a maximal independent subset. In vector spaces bases consist of vectors that span the space, and the theory extends this concept of defining bases in the abstract setup of matroids without referring to vectors or vector spaces. All bases of a given matroid have the same size, a property reminiscent of the dimension in vector spaces, which leads to the matroid invariant known as the rank.

Matroids can be represented in various ways. One common method of representation is through matrices (for vectorial matroids) or adjacency matrices of graphs(for graphic matroid). There is also a geometric lattice, which mirrors the relations between subspaces in a finite dimentional vector space. This duality and interplay between the combinatorial and geometric perspectives are part of what makes Matroid theory intriguing.

# Chapter 1

# Preliminaries

**Definition 1.0.1.** A graph $G$ is a pair $G = (V, E)$ consisting of a finite set $V$ and a set $E$ of 2- element subset of $V$. The elements of $V$ are called vertices (points,nodes) and elements of $E$ are called edges. The set $V$ is known as the vertex set of $G$ and $E$ as the edge set of $G$.

**Definition 1.0.2.** Suppose $\{u, v\}$ is a member of $E(G)$, then we say $u$ and $v$ are joined by an edge in $G$. If we denote by $e$, we can then write $e = uv$ which is an edge of $G$.

**Definition 1.0.3.** An edge that joins a vertex to itself is called a loop.

**Definition 1.0.4.** Let $G$ be a graph with vertex set $V(G)$ and set $E(G)$. A subgraph of $G$ is a graph all of whose vertices belong to $V(G)$ and $E(G)$.

**Definition 1.0.5.** If $d(v) = 0$ then $v$ is known as an isolated vertex.

**Definition 1.0.6.** A path in a graph is a finite or infinite sequence of edges which joins a sequence of vertices.

**Definition 1.0.7.** A planar graph is a graph that can be embedded in the plane, i.e. it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.

**Definition 1.0.8.** A complete graph is a graph in which each vertex is connected to every other vertex.

**Theorem 1.0.1.** Let $u$ and $v$ be nonadjacent vertices in a graph $G$. The minimum number of vertices in a $u - v$ separating set equals the maximum number of internally disjoint $u - v$ paths in $G$.

# Chapter 2

# Definition of Matroids

## 2.1 Independent Sets

**Definition 2.1.1. (Independence Axioms)** Given some finite set $E$, the set system $(E, \mathscr{I})$ is a **matroid** if the following are satisfied:

(1) $\varnothing \in \mathscr{I}$.

(2) If $X \in \mathscr{I}$ and $Y \subseteq X$ then $Y \in \mathscr{I}$.

(3) If $X, Y \in \mathscr{I}$ and $|X| > |Y|$ then there exist $x \in X - Y$ such that $Y \cup \{x\} \in \mathscr{I}$.

We write $M(E, \mathscr{I})$ or simply $M$ if $E$ and $\mathscr{I}$ are self-evident. Axiom (3) will be called the *independence augmentation axiom*. Any set system $(E, \mathscr{I})$ with $\mathscr{I}$ satisfying axioms (1) and (2) will be called an *independence system*. Since any matroid is also an independence system by definition, any future definition on independence systems applies to matroids as well. Consider an independence system $(E, \mathscr{I})$. The members of $\mathscr{I}$ are called *independent* while those of $2^E - \mathscr{I}$ *dependent*. The collection of independent sets for some $X \subseteq E$ will be

$$\mathscr{I}(X) = \{Y \subseteq X : Y \in \mathscr{I}\}.$$

**Theorem 2.1.1.** Suppose that $X, Y$ are sets in a matroid $M(E, \mathscr{I})$ and $|X| > |Y|$. Then there exists some $Z \subseteq X - Y$ such that $|Y \cup Z| = |X|$ and $Y \cup Z \in \mathscr{I}$.

*Proof.* Let $Z \subseteq X - Y$ be a maximal set such that $Y \cup Z \in \mathscr{I}$ and assume that $|Y \cup Z| < |X|$. We know from (3) that such a $Z$ exists, at least for $|Z| = 1$. Since both $X$ and $Y \cup Z$ are independent, there exists some $x \in X - (Y \cup Z)$ such that $(Y \cup Z) \cup x \in \mathscr{I}$. Since $x \notin Z$ it implies that $Z$ is not maximal, a contradiction. □

**Definition 2.1.2. (Graphic Matroids)** A matroid isomorphic to the matroid $M$ $(G)$ with ground set $E = M(G)$ and independence family

$$\mathscr{I} = \{X \subseteq E : G[X] \text{ is a forest}\},$$

for a graph $G$, will be called *graphic matriod.*

**Definition 2.1.3. (Representable Matroids)** A matroid isomorphic to the matroid $M[A]$ with ground set $E = \{\text{set of columns of } A\}$ and independence family

$$\mathscr{I} = \{X \subseteq E : X \text{ is a linearly independent set of vectors in } \mathbb{F}\},$$

for a matrix $A \in \mathbb{F}^{m \times n}$ in some field $\mathbb{F}$, will be called $\mathbb{F}$ *-representable matroid* or simply *representable* .

**Definition 2.1.4. (Transversal Matroids)** A matroid isomorphic to the matroid $(E, \mathscr{F})$ with independence family

$$\mathscr{I} = \{X \subseteq E : X \text{ is a partial transversal of } \mathscr{F}\},$$

for a set system $(E, \mathscr{F})$, will be called *transversal matroid.*

For a given a graph $G$ the matroid $M(G)$ will be called the cycle matroid of $G$, while for a given matrix $A$ the matroid $M[A]$ will be called the vector matroid of $A$. Moreover, $GF(2)$ - representable matroids will be called binary.

**Definition 2.1.5. (Trivial Matroid)** Given any non-empty finite set $E$, we can define on it a matroid whose only set is the empty set $\varnothing$. This matroid is the *trivial matroid* on $E$.

**Definition 2.1.6. (Discrete matroids)**At the other extreme is the*(discrete matroid)* on $E$, in which every subset of $E$ is independent.

**Definition 2.1.7. (Uniform matroids)** The $k$-uniform matroid on $E$, whose bases are those subsets of $E$ with exactly $k$ elements, the trivial matroid on $E$ is $o$-uniform and the discrete matroid is $|X|$ - uniform. [**?**]

**Definition 2.1.8. (Isomorphic matroid )** Two matroid $M_1$ and $M_2$ to be isomorphic if there is a one-one correspondence between their underlying sets $E_1$ and $E_2$ that preserves independence. Thus, a set of elements of $E_1$ is independent in $M_1$ if and only if the corresponding set of elements of $E_2$ independent in $M_2$.

## 2.2 Bases

**Definition 2.2.1. (Bases)** Given an independence system $(E, \mathscr{I})$, the maximal independent sets will be called *bases*. The family of bases will be denoted by $\mathscr{B}$. The collection of bases for some $X \subseteq E$ will be denoted by $\mathscr{B}(X)$ and is defined as

$$\mathscr{B}(X) = \{Y \subseteq X : Y \in \mathscr{I}, Y \cup \{x\} \notin \mathscr{I} \text{ for all } x \in X - Y\}.$$

It follows that $\mathscr{B}(E) = \mathscr{B}$. Note that bases in independence systems can have different cardinalities.

**Lemma 2.2.1.** An independence system $(E, \mathscr{I})$ is a matroid if and only if for any $X \subseteq E$ all bases of $X$ have the same cardinality.

**Theorem 2.2.1. (Basis Axioms)** A collection $\mathscr{B} \subseteq 2^E$ is the set of basis of a matroid $M(E, \mathscr{I})$ if and only if the following are satisfied :

(i) $\mathscr{B} \neq \varnothing$.

(ii) If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 - B_2$ then there exists $y \in B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathscr{B}$.

## 2.3 Circuits

**Definition 2.3.1. (**Circuits) Given an independence system $(E, \mathscr{I})$ the minimal dependent sets will be called *circuits*. The family of circuits will be denoted by $\mathscr{C}$.

The collection of circuits for some $X \subseteq E$ will be denoted by $\mathscr{C}(X)$ and is defined as

$$\mathscr{C}(X)=\{Y \subseteq X : Y \notin \mathscr{I}, Y - \{y\} \in \mathscr{I} \text{ for all } y \in Y\}.$$

A single ton $e \in E(M)$ that is a circuit, will be called a *loop* of $M$.

**Theorem 2.3.1. (Circuit Axioms)** A collection $\mathscr{C} \subseteq 2^E$ is the set of circuits of a matroid $M(E, \mathscr{I})$ if and only if the following are satisfied :

(i) $\varnothing \notin \mathscr{C}$.

(ii) If $C_1, C_2 \in \mathscr{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$.

(iii) If $C_1, C_2 \in \mathscr{C}$, $C_1 \notin C_2$ and $e \in C_1 \cap C_2$ then there exists $C_3 \in \mathscr{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - \{e\}$.

## 2.4 Rank

**Definition 2.4.1. (Rank)** Given an independence system $(E, \mathscr{I})$, the *rank function* $r : 2^E \to \mathbb{Z}_+$ is defined as

$$r(X) = max\{|Y| : Y \subseteq X, Y \in \mathscr{I}\}$$

for any $X \subseteq E$.

**Theorem 2.4.1. (Rank Axioms)** A function $r : 2^E \to \mathbb{Z}$ is the rank function of a matroid $M(E, \mathscr{I})$ if and only if the following are satisfied for all $X, Y \in E$ :

(i) $0 \leq r(X) \leq |X|$.

(ii) If $Y \subseteq X$ then $r(Y) \leq r(X)$.

(iii) $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$.

**Definition 2.4.2. (Low Rank)** Given an independence system $(E, \mathscr{I})$, the *low rank function* $lr : 2^E \to \mathbb{Z}$ is defined as

$$lr(X) = min\{|Y| \subseteq X, Y \in \mathscr{I}, Y \cup x \notin \mathscr{I}, \text{ for all } x \in Y - X\},$$

for any $X \subseteq E$.

## 2.5 Closure

**Definition 2.5.1. (Closure)** Given an independence system $(E, \mathscr{I})$ the *closure operator* is a set function $cl : 2^E \to 2^E$ defined as

$$cl(X) = \{y \in E : r(X \cup \{y\}) = r(X)\}.$$

for any $X \subseteq E$.

**Theorem 2.5.1. (Closure Axioms)** A function $cl : 2^E \to 2^E$ is the closure operator of a matroid $M(E, \mathscr{I})$ if and only if the following are satisfied for all $X, Y \subseteq E$ and $x, y \in E$ :

(i) If $X \subseteq E$ then $X \subseteq cl(X)$.

(ii) If$X \subseteq Y \subseteq E$ then $c(X) \subseteq c(Y)$.

(iii) If $X \subseteq E$ then $cl(cl(X)) \subseteq cl(X)$.

(iv) $X \subseteq E, x \in E, y \in cl(X \cup \{x\} - cl(X)$ then $x \in \text{cl } (X \cup \{y\})$.

# Chapter 3

# Duality

## 3.1 Duality

The notion of duality in matroids is similar to the one in optimization, and it generalizes the concepts of orthogonality in vector spaces, and planarity in graphs. As the next theorem demonstrates, for any matroid $M$ we can define another matroid $M^*$ on the same ground set called the *dual* of $M$, such that independent sets, bases, circuits, rank, and any other property of $M$ have well-defined dual counterparts in $M^*$.

**Theorem 3.1.1.** Given a matroid $M(E, \mathscr{B})$, then

$$\mathscr{B}^* = \{X \subseteq E : \text{there exist a base } B \in \mathscr{B}, \text{such that } X = E - B\},$$

is the family of bases of a matroid $M^*$ on $E$, called the *dual* of $M$.

**Theorem 3.1.2.** For a matroid $M$ and $X \subseteq E(M)$

$$r^*(X) = |X| - r(M) + r(E - X).$$

**Theorem 3.1.3.** If $G$ is a planar graph with geometric dual $G^*$ then $M^*(G) = M(G^*)$.

**Theorem 3.1.4.** $G$ is planar if and only if $M(G) = M^*(G)$.

Therefore, the class of graphic matroids is not dual-closed, and the duals of graphic matroids which are not graphic will be called *cographic*. Those matroids which are both graphic and cographic are called *planar*.

## 3.2 Minors

**Definition 3.2.1. (Contraction)** : Let $M$ be a matroid on $E$. The two fundamental operations on $M$ that we have introduced so far are deletion and the taking of duals. The next definition combines these two operations. Let $M/T$ , the *contraction* of $T$ from $M$, be given by $M/T = (M^*/T)^*$.

Evidently $M/T$ has ground set $E - T$. We shall sometimes write $M(E - T)$ for $M/T$ and call it the contraction of $M$ onto $E - T$.

**Result 1.** $M(G/T) = M(G)/T$

**Proposition 3.2.1.** If $G$ is a graph, then $M(G)/T = M(G/T)$ for all subsets $T$.

*Proof.* We shall show that, for every edge $e$ of $G$,

$$M(G)/e = M(G/e).$$

The proposition then follows by a routine induction argument on $|T|$. If $e$ is a loop of $G$, then $G/e = G/e$ and $M(G)/e = M(G)/e$. The result follows in this case by result 1. Now suppose that $e$ is not a loop of $G$. Then, for a subset $I$ of $E(G) - e$, it is not difficult to check that $I \cup e$ contains no cycle of $G$ if and only if $I$ contains no cycle of $G/e$. Hence $\mathcal{I}(M(G)/e) = \mathcal{I}(M(G/e))$ and $M(G)/e = M(G/e)$ holds. □

**Corollary 3.2.1.** Every minor of a graphic matroid is graphic.

**Result 2.** Let $A$ be a matrix over a field $\mathbb{F}$ and $T$ be a subset of the set $E$ of column labels of $A$. We shall denote by $A/T$ the matrix obtained from $A$ by deleting all the columns whose labels are in $T$. Evidently

$$M[A]/T = M[A/T].$$

**Proposition 3.2.2.** Every minor of an $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable.

*Proof.* By result 2, every deletion of an $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable. As the dual of an $\mathbb{F}$-representable matroid is also $\mathbb{F}$-representable, we deduce, from the definition of contraction, that every contraction of an $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable. Hence so is every minor. □

**Proposition 3.2.3.** Every minor of a regular matroid is regular.

## 3.3 Connectivity

**Definition 3.3.1. (Seperator)** For a matroid $M(E, \mathscr{C})$ a set $X \subseteq E$ called a *seperator* of $M$ if any circuit $\mathrm{C} \in \mathscr{C}$ is contained in either $X$ or $E - X$.

**Proposition 3.3.1.** For a matroid $M(E, \mathscr{C})$ some set $X \subseteq E$ is a seperator of $M$ if and only if $r(X) + r(E - X) = r(E)$.

**Corollary 3.3.1.** Given a matroid M, a set $X \subseteq E(M)$ is a seperator of $M$ if and only if $M/X = M/X$.

**Proposition 3.3.2.** Given a matroid M, a set $X$ is seperator of $M$ if and only if $X$ is a seperator of $M^*$.

*Proof.* By corollary and proposition, a set $X$ is a seperator of $M$ if and only if

$$M/X = M/X \iff (M/X)^* = (M/X)^* \iff M^*/X = M^*/X,$$

if and only if $X$ is a seperator of $M^*$. □

**Corollary 3.3.2.** A matroid $M$ is connected if and only if $M^*$ is connected .

**Definition 3.3.2. (Matroid $k$ - connectivity)** For a matroid and postive intrger $k$, a partion $X, Y$ of $E(M)$ is a $K$ - seperation of M if

(1) $min\{|X|, |Y|\} \geq k$.

(2) $r(X) + r(Y) \leq r(M) + k - 1$.

The *connectivity number* of matroid is defined as

$$\lambda(M) = min\{k : M \text{ has a } k \text{ - seperation for } k \geq 1\}.$$

while if $M$ does not have a $k$ - seperation for any number $k \geq 1$ then $\lambda(M) = \infty$. We say that a matroid $M$ is $k$ - connected for any $1 \leq k \leq \lambda(M)$. If $\{X, E - X\}$ is a 1 - seperation of a matroid $M$ then by definition we have that $r(X) + R(E - X) - r(M) \leq 0$, and by the submodularity of the rank function $r(X) + r(E - X) - r(M) \geq 0$, which by proposition 2.3.1 means that $X$ is a seperator of $M$. So 1 - seperations are seperators, which implies that a matroid is connected if and only if it is 2 - connected. As it was the case with matroid connectivity, $k$ - connectivity in matroids is also duality invariant.

**Proposition 3.3.3.** For a matroid $M$,$\{X, Y\}$ is a $k$ - seperation of $M$ if and only if $M\{X, Y\}$ is a $k$ - seperation of $M^*$.

*Proof.* Let $X, Y$ be a $k$ - seperation of $M$. From theorem 2.1.2 and since $r(X) + R(E - X) \leq r(M) + k - 1$ by (2) in definition 2.3.2 we have

$$\begin{aligned} r^*(X) + r^*(E - X) &= |X| - r(M) + r(E - X) + |E - X| - r(M) + r(X) \\ &= |E| + r(X) + r(E - X) - 2r(M) \\ &\leq |E| - r(M) + k - 1 \\ &= r^*(M) + k - 1. \end{aligned}$$

Letting $\{X, Y\}$ be a $k$ - seperation of $M^*$ the above computation also applies for the rank of $M$ since the statement of theorem 2.1.2 is self- dual. □

**Corollary 3.3.3.** For a matroid $M$ we have $\lambda(M) = \lambda(M^*)$.

The next theorem states that $k$ - connectivity in matroids is indeed a generalization of $k$ - connectivity in graphs.

**Theorem 3.3.1.** If $G$ is a connected graph then $\lambda(G) = \lambda(M(G))$.

**Definition 3.3.3. (Matroid vertical $k$ - connectivity)** For a matroid $M$ and a positive interger $k$, a partition $\{X, Y\}$ of $E(M)$ is a *vertical $k$ - seperation* of $M$ if

(i) $min\{r(X), r(Y)\} \geq k$,

(ii) $r(X) + r(Y) \leq r(M) + k - 1$.

The *vertical connectivity number* of matroid $M$ is defined as

$$\lambda(M) = min\{k : M \text{ has a vertical } k \text{ - seperation for } k \geq 1\},$$

while if $M$ does not have a vertical $k$ - seperation for any number $k \geq 1$ then $k(M) = \infty$. We say that a matroid $M$ is vertical $k$ - connected for any $1 \leq k \leq k(M)$.

Since $|X| \geq r(X)$ for any $X \subseteq E(M)$ is a matroid $M$, a vertical $k$ - seperation in $M$ induces a $k$ - seperation. Hence, if a matroid is $k$ - connected then it is also vertical $k$ - connected. vertical $k$ - connectivity in matroids generalizes $k$ - vertex - connectivity in graphs, however it is not duality invarient.

# Chapter 4

# Series - Parallel and Delta - Wye Constructions

This chapter is the first of three on matroid tools. Here we construct graphs and binary matroids with elementry procedures. For graphs, the constructions involve adddition of a parallel edge, or subdivision of an edge into two series edges, or substitution of a triangle by a 3-star, or substitution of a 3-star by a triangle. We call the first two operations series-parallel extension steps, for short SP extension steps. Either one of the triangle/3-star substitution steps is a delta-wye step, for short $\delta Y$ step. These operations have a natural translation to operations on binary matroids. The power of SP extension steps is quite limited. Suppose in the graph case one starts with a cycle with just two edges and applies SP extension steps. Then rather simple graphs are produced. They are usually called series-parallel graphs, for short SP graphs. In the binary matroid case, let us start with a circuit containing just two parallel elements. Then we produce nothing else but the graphic matroids of the SP graphs. The situation changes dramatically when we mix SP extension steps with $\delta Y$ steps. In the graph case, suppose we start again with a cycle with two edges. Then we produce all 2-connected planar graphs and more. How much more is a difficult open question. Similarly, suppose that in the binary matroid case we start with a circuit with two edges. Then we produce the graphic matroids of the just described graphs, as well as nongraphic binary matroids. Here too, the class of matroids so obtained is not well understood. We know that every matroid of that class is regular.

## 4.1 Series-Parallel Construction

Start with the cycle with just two edges. In that small graph, replace one edge by two parallel edges or by two series edges. To the resulting graph apply either one of these two operations to get a third graph, and so on.

**Lemma 4.1.1.** Every SP graph is 2-connected and planar. Any minor of an SP graph is also an SP graph, provided the minor has at least two edges and is 2-connected.

*Proof.* The cycle with two edges is 2 -connected and planar. An SP extension step in a graph with at least two edges cannot introduce a 1-separation or destroy planarity. By induction, the SP graphs are 2-connected and planar. For the proof of the second part, paint in a given SP graph the edges of a given 2-connected minor red. Reduce the SP graph by SP reduction steps until the cycle with two edges is obtained. We examine a single reduction step and apply induction. We must consider two cases for that step: deletion of a parallel edge, and contraction of one of two edges with a common degree 2 endpoint. Consider the deletion case. If both edges are red, then the reduction is also an SP reduction in the minor. If exactly one edge is not red, then that edge is deleted. The minor must still be present, since contraction of that edge would turn the red edge into a loop, contrary to the assumption that the minor is 2-connected. If both edges are not red, then the minor is still present after deletion of one of these edges. For if both edges must be contracted to produce the minor, then the second contraction involves a loop, and thus is a deletion. The contraction case is handled analogously.

□

Recall that $K_4$ is the complete graph on four vertices.

**Lemma 4.1.2.** Every 3-connected graph $G$ with at least six edges has a $k_4$ minor.

*Proof.* Take any cycle $C$ of $G$ of minimal length. Since $G$ is 3-connected and has at least six edges, it must have a node that does not lie on $C$. By Menger's Theorem, there are three internally node-disjoint paths from that additional node to three distinct nodes of $C$. Suitable deletions and contractions eliminate all other edges and reduce the cycle and three paths to a $k_4$ minor. □

**Lemma 4.1.3.** $K_4$ is not an SP graph.

*Proof.* $K_4$ does not have series or parallel edges. □

**Lemma 4.1.4.** No SP graph has a $K_4$ minor.

*Proof.* Presence of a $K_4$ minor would contradict Lemmas (3.1.1) and (3.1.3). □

**Theorem 4.1.1.** A 2-connected graph is an SP graph if and only if it has no $K_4$ minor.

*Proof.* The "only if" part is handled by Lemma (3.1.4). We thus prove the converse. Let $G$ be a 2-connected graph without $K_4$ minors. Simple checking validates the small cases with up to five edges. So assume G has at least six edges. By Lemma (3.1.2), $G$ must be 2-separable. Choose a 2-separation so that for the two corresponding graphs $G_1$ and $G_2$, we have $G_1$ with minimal number of edges. Suppose $G_1$ has exactly two edges. These edges must be parallel or incident at a degree 2 node of $G$. Thus, we can reduce and apply induction.Suppose $G_1$ has at least three edges. Let $k$ and $l$ be the nodes of $G_1$ that must be identified with two nodes of $G_2$ to produce $G$. Suppose $G_1$ has an edge $z$ connecting $k$ and $l$. That edge can be shifted from $G_1$ to $G_2$. The corresponding new 2-separation contradicts the minimality assumption onthe edge set of $G_1$. Similarly, the nodes $k$ and $l$ cannot have degree 1 in $G_1$. Add an edge $e$ to $G_1$ connecting nodes $k$ and $l$. The new graph $G_1'$ is isomorphic to a proper minor of $G$. By induction, $G_1'$ is an SP graph. By the above discussion, in $G_1'$ the edge $e$ is not parallel to another edge, and it does not have an endpoint of degree 2. Thus, any SP reduction step in $G_1'$ can be carried out in $G$ as well. We perform one such step in $G$, and invoke induction for the reduced graph. □

**Lemma 4.1.5.** An SP graph without parallel edges either is a cycle with at least three edges, or has two internally node-disjoint paths with the following properties. Each path has at least two edges. Each intermediate node of the two paths has degree 2 in the graph, while the endpoints have degree of atleast 3.

**Corollary 4.1.1.** An SP graph with at least four edges and without parallel edges has at least two nonadjacent nodes with degree 2.

*Proof.* If the SP graph is a cycle, then the conclusion is immediate. So assume that the SP graph is not a cycle. Then each one of the two paths postulated in Lemma (3.1.5) has at least one intermediate degree 2 node. Thus, the graph has two nonadjacent degree 2 nodes. □

We introduce two interesting subclasses of the class of SP graphs by excluding certain graphs as minors. One of the excluded graphs we already know. It is $K_{2,3}$, the complete bipartite graph with two vertices on one side and three on the other one. The second excluded graph is the *double triangle*, obtained from the triangle by replacing each edge by two parallel edges. We denote that graph by $C_3^2$. We want to characterize first the SP graphs without $K_{2,3}$ minors, and then those without $C_3^2$ minors. To this end, define a graph to be*outer planar* if it that all vertices lie on the infinite face can be drawn in the plane so that all vertices lie on the infinite face.

**Theorem 4.1.2.** The following statements are equivalent for a 2-connected graph $G$ with at least two edges.

(a) $G$ has no $K_4$ or $K_{2,3}$ minors.

(b) $G$ is an SP graph without $K_{2,3}$ minors.

(c) $G$ is outerplanar.

*Proof.* By Theorem (3.1.1), $G$ is an SP graph if and only if it has no $K_4$ minors. Thus, $(a) \iff (b)$. To show $(b) \iff (3)$, let $G$ be an SP graph without $K_{2,3}$ minors. Define $C$ to be a cycle of $G$ of maximum length. Suppose $G$ has a node $v$ that does not lie on $C$. Since $G$ is 2-connected,there exist two paths from node $v$ to distinct nodes $i$ and $j$ on $C$ so that these paths have only the node $v$ in common. If $i$ and $j$ are connected by an edge of $C$, then $C$ can be extended to a longer cycle using the two paths,a contradiction. If $i$ and $j$ are not joined by an edge of $C$, then $C$ and the two paths are easily reduced to a $K_{2,3}$ minor of $G$, another contradiction. Thus, all nodes of $G$ occur on $C$. For the proof of outerplanarity, we may assume that $G$ has no parallel edges. Draw $C$ in the plane, say using a circle. Then draw the remaining edges, each time placing the edge inside the circle as a straight line segment. If any two such edges cross, then these edges plus $C$ can be reduced to a $K_4$ minor of $G$, a contradiction. Thus, no edges cross, and we have produced an outerplanar drawing of $G$. For (3) $\iff$ (1), we note that $K_4$ and $K_{2,3}$ are not outerplanar and that outerplanarity is maintained under minor-taking. □

## 4.2 Delta-Wye Construction for Graphs

The simplicity of SP graphs gives way to far more complicated graphs when we permit two operations in addition to SP extensions. One of them is the replacement of a triangle by a 3-star, and the second one is the inverse of that step. Either operation we call a $\Delta Y$ exchange. Define a sequence of SP extensions and $\Delta Y$ exchanges to be a $\Delta Y$ extension sequence. The inverse sequence is a $\Delta Y$ reduction sequence. A 2-connected graph is $\Delta Y$ reducible if there is a $\Delta Y$reduction sequence that converts the graph to a cycle with just two edges. In this section, we show that $\Delta Y$ extension sequences applied to such a cycle create all 2-connected planar graphs and more. Any graph so producible is a $\Delta Y$ graph.

As an example for $\Delta Y$ reduction sequences, we reduce $K_5$, the complete graph on five vertices, to the cycle with two parallel edges. Series or parallel edges of SP reductions. We know that $M(K_5)$ is not cographic. Thus, $K_5$ is nonplanar, and the example demonstrates that $\Delta Y$ graphs may be nonplanar. A $\Delta Y$ exchange may not preserve 2-connectedness. For example, when a vertex of a triangle in a 2-connected graph has degree 2, then replacement of that triangle by a 3-star produces a 1-separable graph. The next lemma gives the conditions under which 2-connectedness is maintained.

In each graph of the reduction sequence, the triangle or 3-star involved in a $\Delta Y$ exchange is indicated by bold lines. Similarly, we emphasize the series or parallel edges of SP reductions. We know that $M(K_5)$ is not cographic. Thus, $K_5$ is nonplanar, and the example demonstrates that $\Delta Y$ graphs may be nonplanar. A $\Delta Y$ exchange may not preserve 2-connectedness. For example, when a vertex of a triangle in a 2-connected graph has degree 2, then replacement of that triangle by a 3-star produces a 1-separable graph. The next lemma gives the conditions under which 2-connectedness is maintained.

**Lemma 4.2.1.** Let $G$ be a 2-connected graph. Then a triangle to 3-star exchange (resp. 3-star to triangle exchange) in $G$ produces a 2-connected graph $G'$ if and only if the triangle (resp. 3-star) does not contain two edges in series (resp. in parallel).

The asymmetry of arguments in the proof of Lemma (3.2.1) is due to the fact that a triangle is always a cycle of a graph, while a 3-star is not always a cocycle. Note that the conditions of Lemma (3.2.1) are automatically satisfied in $\Delta Y$ reduction sequences where $\Delta Y$ exchanges are done only when an SP reduction is not possible. Our goal is to show that the class of $\Delta Y$ graphs includes all 2-connected planar graphs. That goal is restated in the next theorem.

**Theorem 4.2.1.** Every 2-connected planar graph is $\Delta Y$ reducible.

**Lemma 4.2.2.** If a 2-connected graph or plane graph G is $\Delta Y$ reducible, then every 2-connected minor $H$ of $G$ is $\Delta Y$ reducible as well.

**Lemma 4.2.3.** Every plane graph is a minor of some grid graph.

*Proof.* (Sketch) We may assume that the given plane graph is 2-connected, since this can be achieved by the addition of edges. Split each vertex of that plane graph so that a 2-connected plane graph $G$ results where the degree of each vertex is at most 3. By a suitable subdivision of edges, $G$ can be embedded into a grid graph as follows. First embed any one face of $G$, but not the outer one. Then embed one face at a time so that each one of the successive subgraphs of $G$ so embedded is 2-connected. □

# Chapter 5

# Convex Polytypes associated with Matroids

## 5.1 Convex polytopes and linear programming

If $S$ is a finite set, we let $\mathbb{R}^s(\mathbb{R}^s_+)$ denote the space of real valued (non negative) row vectors with coordinates indexed by $S$. For each $x \in \mathbb{R}^s$ and $e \in S$ denote the $e^{th}$ coordinate of $x$ by $x(e)$. For $x, y \in \mathbb{R}^s$ we write $x \geq y$ if $x(e) \geq y(e)$ for $\forall e \in S$, and call $y$ a *subvector* of $x$. This induces a partial order on $\mathbb{R}^s$ and $<$, $\leq$, $>$ are now defined in the obvious way. For $x \in \mathbb{R}^s$ and $A \subseteq S$ we define

$$x(A) = \Sigma_{e \in A} x(e)$$

and call the modulus $|x|$ of $x$ the quantity

$$|x| = x(S) = \Sigma_{e \in S} |x(e)|.$$

For $x, y \in \mathbb{R}_+^S$ and $e \in S$, we define

$$(x \wedge y)(e) = min(x(e), y(e))$$
$$(x \vee y)(e) = max(x(e), y(e))$$

and thus define $x \wedge y$, $x \vee y$ is obvious way.

A subset $X \subseteq \mathbb{R}^S$ is convex if $x, y \in X$ implies that for all $\lambda, 0 \leq \lambda \leq 1$,

$$\lambda x + (1 - \lambda)y \in X.$$

The convex hull of $X$ , denoted by $co(X)$ is the intersection of all convex sets contatining $X$, or alternatively the smallest convex set containing $X$. A hyperplane of $\mathbb{R}^S$ is a subset $H \subseteq \mathbb{R}$ such that $\exists c \in \mathbb{R}^S/0, d \in \mathbf{R}$ such that

$$H = \{x \in \mathbb{R}^S : cx' = d\}.$$

Clearly $H$ is uniquely determined by the pair $(c, d)$ and is a maximal proper affine subspace of $\mathbb{R}^S$. Its equation is

$$cx' = d.$$

Associated with such a hyperplane are two half spaces

$$H^+ = \{x : cx' \geq d\}$$
$$H^- = \{x : cx' \leq d\}.$$

Clearly a half space is closed and convex .Thus the intersection of half spaces is closed and convex .

**Definition 5.1.1.** A convex polytope is a bounded region of $\mathbb{R}^S$ which can be expressed as the intersection of a finite number of half spaces. Given a polytope $P \subseteq \mathbb{R}^S$ which is defined by

$$P = \{X \in \mathbb{R}^S : C_i x' \leq d_i : i \in T\}$$

We call the face associated with $J \subseteq T$, the subset $P(J)$ of $P$ defined by

$$P(J) = \{x \in P : c_i x' = d_i, i \in J, c_i x' < d_i, i \in T/J\}.$$

The rank of the face $P(J)$ is defined by $|S| - d(C^J)$ where $d(C^J)$ is the rank of the matrix $C^J$ which has row vectors $\{c_i : i \in J\}$. A vertex of $P$ is a face of zero rank. Thus to find the vertices of the convex polytope $P$ we would have to solve the equations

$$c_i x' = d_i \ ; i \in J$$

for all $J \subseteq T$ such that $|J| = |S|$ and then check which of these solutions were members of $P$. [2]

**Theorem 5.1.1.** A convex polytope $P$ has only a finite number of vertice; moreover it can be expressed as the convex hull of its set of vertices. Conversely given any finite set $X$ of points of $\mathbb{R}^S$ the convex hull of $X$ is a convex polytope with vertex set a subset of $X$. This convex polytope can be expressed in the form $\alpha$ where the hyperplanes $c_i x' = d_i$ are its maximal proper faces.

We preset the bare outlines of linear programming theory. Let $S, T$ be finite sets. Let $A = \{a_{i,j} : i \in T, j \in S\}$ be a matrix with $a_{i,j} \in \mathbb{R}$.Let $b \in \mathbb{R}^T, c \in \mathbb{R}^S$. A (primal) linear programme is

$$\text{maximize } cx'$$

for $x \in \mathbb{R}^S$ satisfying

$$Ax' \leq b'$$
$$x \geq 0$$

The dual linear programme is

$$\text{minimize } by'$$

for $y \in \mathbb{R}^T$ satisfying

$$A'y' \geq c'$$
$$y \geq 0.$$

A vector $x$ satisfying (1) and (2) is a feasible solution to the primal problem. A vector $y$ satisfying (3) and (4) is a feasible dual solution. A feasible primal solution which maximizes $cx'$ is an optimal primal solution. An optimal dual solution is defined analogously.

**Theorem 5.1.2.** For any linear programming maximization problem exactly one of the following situations occurs.

(1) There exists no feasible solution.

(2) For any $\alpha \in \mathbb{R}$ there is a feasible solution $x$ such that $cx' > \alpha$.

(3) There is an optimal (feasible) solution.

**Theorem 5.1.3.** If $x$ is a feasible primal solution and $y$ is a feasible dual solution then

$$cx' < by'.$$

**Theorem 5.1.4.** If there is a feasible primal solution and an upper bound for $cx'$ over all feasible primal solutions $x$, then there is an optimal primal solution $u$ and an optimal dual solution $v$ and $cu' = bu'$.

**Theorem 5.1.5.** (1) Let $P$ be a convex polytope in $\mathbb{R}^S$. Then for any $c \in \mathbb{R}^S$ there exists a vertex $v$ of $P$ which maximizes $cx'$ over $P$.

(2) Let $P$ be a covex polytope in $\mathbb{R}^S$ and let $v$ by any vertex of $P$.Then there exist $c \in \mathbb{R}^S$ such that $v$ is the unique member of $P$ maximizing $cx'$ over $P$.

**Definition 5.1.2.** A polymatroid $\mathbb{P}$ is a pair $(S, P)$ where $S$, the ground set is a non- empty, finite set and $P$, the set of independent vectors of $\mathbb{P}$ is a non-empty compact subset of $\mathbb{R}^S_+$ such that :

(1) every subvector of an independent vector is independent,

(2) for every vector $a \in \mathbb{R}^S_+$, every maximal independent subvector $x$ of a has the same modulus $r(a)$, the vector rank of $a$ in $\mathbb{P}$.

In the above definition "maximal" has its obvious meaning that there exists no $y > x$ having the properties of $x$; since $P$ is compact it is well defined.

**Definition 5.1.3.** A polymatroid $\mathbb{P}$ is a pair $(S, P)$ where $S$ the ground set is a non- empty finite set and $P$, the set of independent vectors of $\mathbb{P}$ is a non-empty compact subset of $\mathbb{R}^S$ such that (1) holds and also: If $u, v \in P$ and $|v| > |u|$, then there is a vector $w \in P$ such that

$$u < w \leq u \wedge v$$

**Definition 5.1.4.** A polymatroid $\mathbb{P}$ is a pair $(S, \rho)$ where $S$ , the ground set is a non-empty finite set and $\rho$, the ground set rank function is a function: $2^S \to \mathbb{R}^+$ satisfying

$$A \subseteq B \subseteq S \implies \rho A \leq \rho B$$

$$A, B \subseteq S \implies \rho A + \rho B \geq \rho(A \cup B) + \rho(A \cap B)$$

$$\rho(\phi) = 0$$

and the vectors $x \in \mathbb{R}_+^S$ such that $x(A) < \rho A$ for all $A \subseteq S$ are the independent vectors of $\mathbb{P}$. For each vector $a$ in $\mathbb{R}_+^S$, the vector rank $r(a)$ of $a$ is given by

$$r(a) = min(a(X) + \rho(S/X) : X \subseteq S).$$

**Lemma 5.1.1.** The vector rank function $r$ of Definition 1 satisfies for all $u, v \in \mathbb{R}^S$,

$$r(u) + r(v) \geq r(u \vee v) + r(u \wedge v).$$

*Proof.* Let $a$ be a base of $u \vee v$. By (2), $\exists$ $b$, independent in $P$, satisfying

$$a \leq b \leq u \vee v$$

and

$$r(b) = |b| = r(u \vee v)$$

since $a = b \wedge (u \wedge v)$ we get

$$a + b = b \wedge u + b \wedge v$$

but $b \wedge u, b \wedge v$ are indepentent subvectors of $u, v$ respectively. Thus

$$\begin{aligned} r(u \wedge v) + r(u \vee v) &= |a| + |b| \\ &= |b \wedge u| + |b \wedge v| \\ &= r(u) + r(v). \end{aligned}$$

□

## 5.2 Polymatroids and Submodular Set Functions

we know that if $\mathbb{P} = (S, P, \rho)$ is a polymatroid in $\mathbb{R}^s$, its independence polytope is the region of $\mathbb{R}^S$ defined by the intersection of the half spaces

$$x(A) \leq \rho A \ , \ A \subseteq S.$$

We also know that $\rho$ is a non-decreasing submodular set function. In this section we prove conversely that polymatroids can be obtained in this way from arbitrary non-negative submodular functions.

**Theorem 5.2.1.** Let $\mathscr{L}$ be a lattice of subsets of the finite set $S$ ordered by inclusion and with the lattice operation of meet, set intersection. Then if $\mu : \mathscr{L} \to \mathbb{R}^+$ is submodular on $\mathscr{L}$, that is

$$\mu(A \vee B) + \mu(A \wedge B) \leq \mu A + \mu B \ , \ A, B \in \mathscr{L},$$

the region $P(S, \mu)$ of $\mathbb{R}^S$ defined by the intersection of the half spaces

$$x(A) \leq \mu A; A \ in \mathbb{Y} \tag{5.1}$$

$$x \geq 0, \tag{5.2}$$

is the independence polytope of a polymatroid $\mathbb{P}$.

**Example 5.2.1.** Let $S = \{1, 2, 3\}$. Let be the lattice of subsets $\phi$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2, 3\}$; and let $u : L \to R$ be defined by $\mu(\phi) = 0$, $\mu\{1\} = 3$, $\mu\{2\} = 5$, $\mu\{3\} = 4$, $\mu\{1, 2, 3\} = 6$.Then $\mu$ is submodular on $\mathscr{L}$ and thus the polyhedron

$$X_1 \leq 3$$

$$X_2 \leq 5$$

$$X_3 \leq 4$$

$$X_1 + X_2 + X_3 \leq 6$$

$$X_i \leq 0$$

is a polynomial in $\mathbb{R}^3$. [2]

**Corollary 5.2.1.** The rank function of the polymatroid $\mathscr{L} = P(S, \mu)$ is given for all $a$ in $\mathbb{R}_+^S$ by

$$r(a) = min_{X \in \mathscr{L}}(\mu X + a(S/X)).$$

*Proof.* Let $\alpha \in \mathbb{R}_+^S$ and let $x \in P(S, \mu) \cap C(a)$. Then for all $A \in \mathscr{L}$,

$$|x| = x(A) + x(S/A) \leq \mu(A) + a(S/A).$$

Hence since $r(a) = |x|$ for some $x$,

$$r(a) \leq min_{A \in \mathscr{L}}(\mu A + a(S/A)).$$

Suppose we have strict inequality in this equation and let $y \in P(S,\mu) \cap C(a)$ be such that $|y| = r(a)$. Define $\mathscr{L}(y)$ as above , if $S \in \mathscr{L}(Y)$, then $|y| = \mu S$ contracting the above strict inequality . Hence $D \neq S$. Now $\forall X \in \mathscr{L}$

$$y(S) < r(a) < \mu(X) + a(S/X).$$

In particular

$$y(S) < \mu(D) + a(S/D)$$

and so

$$y(S/D) < a(S/D).$$

Hence there must exist $e \in S/D$ such that $y(e) < a(e)$. If we augment $y$ to $z$ then $z$ will belong to $P(S,\mu)$ and $C(a)$ contradicting our choice of $y$. Hence we cannot have the above strict inequality and the result follows. □

# Conclusion

The conclusion of Matroid Theory often focuses on its broad applicability and the significant impact it has had on both pure and applied mathematics, including areas like combinatorics, graph theory, algorithm design, and optimization. Matroid theory provides a rich, unifying framework that captures the essence of independence in a very general setting. In summary, Matroid Theory provides a deep and coherent structure for understanding and formalizing the concept of independence, which is present in mathematical settings. Matroids abstract the essential properties of linear independence in vector spaces and mutual exclusivity in set systems, thus offering a powerful toolkit for generalizing and transferring results across different mathematical domain. This project embarked on a comprehensive journey through the intricate landscape of matroid theory, from its fundamental definitions to its profound implications in various domains of mathematics and its applications. We began by establishing the basic concepts of independence, circuits, bases, and rank functions, laying the groundwork for understanding the structure of matroids. The concept of matroids has led to efficient algorithm design for various problems, making the theory directly applicable to real-world scenarios where resource optimization is critical.

As we continue to develop Matroid Theory, we unlock further potential, particularly with the advent of computational tools that allow for the exploration of complex matroids that were previously inaccessible. The study of matroids is a testament to the beauty of mathematical abstraction and its potential to solve concrete problems, allowing mathematicians and scientists to harness the power of structure and symmetry in diverse situations. Indeed, as our understanding of matroids expands, so too does our ability to shape the algorithms and technologies of the future. Matroid Theory, therefore, stands not only as a significant achievement in the field of mathematics but also as a beacon guiding future explorations into the realms of complexity and computation.

# References


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# Ramsey Theory

Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics

by


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**2024**

# CERTIFICATE

This is to certify that the project entitled **“Ramsey Theory”** submitted to Postgraduate and Research Department of Mathematics in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics, is a bonafide record of project work done by **Ms. CHRISTEENA MARIYA FRANCIS (CCAWMMS010)** during the period of her study in the Postgraduate and Research Department of Mathematics, Christ College (Autonomous), Irinjalakuda, under my supervision and guidance during the year 2023-2024

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Date : 25 March 2024

# DECLARATION

I hereby declare that the project work entitled "**Ramsey Theory**" submitted to Christ College(Autonomous), Irinjalakuda in partial fulfilment of the requirement for the award of Master Degree of Science in Mathematics is a record of project work done by me during the period of my study in the Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda.

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Date : 25 March 2024

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**Christeena Mariya Francis**

# Contents

# List of Figures

# Chapter 1

# Introduction

In this chapter we will see the invention of Ramsey Theory

This project is based on the theory proposed by Ramsey. This is fascinating branch of mathematics that studies the emergence of order in seemingly chaotic structures. Ramsey Theory is named after *Frank Plumpton Ramsey* who did seminal work in this area before his untimely death at age 26. His paper was "On a Problem of Formal Logic". He was a British philosopher, mathematician and economist who made important contributions to logic, philosophy of mathematics and decision theory during his short life. Ramsey's work on logic laid the foundation for modern decision theory, and his ideas deeply influenced later philosophers and economists. His name is particularly associated with the words and quot; Ramsey theory and quot. It is a branch of mathematical field of combinatorics that focuses on the appearance of order in a substructure given a structure of a known size.

We can see Ramsey Theorem in different ways, *Ramsey Theorem* states that "for any large enough graph, there is an independent set of size $s$ or a clique of size

$t$".

In general terms, "For all positive integers $k, l$ there exist $R(k, l)$ such that if $N >= R(k, l)$ and the edges of $KN$ are coloured Red and Blue the either there is a 'Red $k$-clique' or there is a 'Blue $l$-clique' ".

Also state that " Given any positive integers $p$ and $q$, their exist a smallest integer $n = R(p, q)$ such that every 2- colouring of the edges $K_n$ contains either a complete subgraph on $p$ vertices, all of whose edges are in 1 colour, or a complete subgraph on $q$ vertices, all of whose edges are in colour 2".

Ramsey posed a question related to combinatorics and probability, which eventually led to the development of Ramsey theory. The problem can be stated as follows: Given six people at a party, some of whom are friends and some of whom are not, what is the minimum number of people needed to ensure that there are either three mutual friends or three mutual strangers? We can explain like this.

## 1.1 Real life Example of Ramsey Theory

The classical problem in Ramsey Theory is party porblem, which asks the minimum number of guests $R(m, n)$ that must be invited so that at least $m$ will know each other (i e, there exists a clique of order $m$) or at least $n$ will not know each other ( i e, there exist an independent set of order $n$)

There is a party of six people, either there is are three who all know each other (mutual friends) or there are three none of whom knows either of other two (mutual strangers). We can solve by it graph theory.

Consider the complete graph $G = K_6$. Represent the six people by the six vertices of G. Any two vertices $a$ and $b$ are defined to be adjacent, if and only if the corresponding persons know each other. Now colour their edges by blue. If the person don't know each other, colour their edges between the corresponding vertices by red.

If there are three people who know each other then this is represented by blue triangle in $K_6$. Similarly, if there are three people who don't know each other then this is represented by a red triangle.

Take an arbitrary vertex a . Its degree is five in $K_6$.When we colour edges incident with a either blue or red. So one colour must be used at least three times. In this case it is a monochromatic triangle. We will discuss monochromatic in later. [1]

## 1.2 Outline of Project

In **Chapter 1** we will discuss about intoduction about Ramsey Theory. We can see reason for invention of Ramsey theory in this chapter. We can explain it through Graph Theory

In **Chapter 2** is literature review. In this chapter we will see the books and journal papers that are supportive documents for the project.

In **Chapter 3** we will see the definition related to graph theory and definitions related to Ramsey Theory. We can see the examples which supporting the definition and also some counter example.

In **Chapter 4**, we discuss some important theorems that are using in Ramsey Theory.

In **Chapter 5**, we conclude the study by discussing the application of Ramsey theory.

# Chapter 2

# Literature Review

**[1] John Clark and Derek Allan Holton, *A First Look at Graph Theory*, 1991**

It is an excellent introductory book that offers a gentle introduction to graph theory, suitable for beginners in mathematics or computer science, it published by World Scientific Publishing Co. Pte. Ltd. in 1991. He use simple example to explain the Ramsey theory. Fist he started with party problem. Solved by graph. He used party problem with 6 members. In graph it become 6 vertices. The members who know each other by blue edge and members who does not know each other coloured by red. It give monochromatic triangle. He gives definition to monochromatic. He gave examples to generalised party problem also.

**[2] Fred S. ROBERTS* *Application of Ramsey Theory*, 1983**

In his paper we can see some application related to Ramsey theory. It will help us know about Ramsey theory in detailed way.

**[3] Douglas B. West, *Introduction to Graph Theory*, 2nd edition, 2000**

In second edition he explained clearly about Ramsey number and also graphs in general terms. Let $r$ and $p_1, \cdots p_k$ be positive integers. If there is a $N$ such that every $k$-coloring of $\binom{[N]}{r}$yields an i-homogeneous set of size $p_i$ for some i, then the smallest such integer is called **Ramsey number** $R(p_1, \cdots p_k : r)$. Ramsey theorem states that we can find such integer for every choice of $r$ and $p_1, \cdots p_k$. He explain it by some simple examples. And he gives the definition of Ramsey Graph. Given simple graphs $G_1, \cdots G_k$, the (graph) Ramsey number $R(G_1, \cdots G_k)$ is the smallest integer $n$ such that ever $k$-coloring of $E(K_n)$ contains a copy of $G_i$ in color $i$ for some $i$. He states the Ramsey Theorem that given positive integers $r$ and $p_1, \cdots p_k$ there exist an integer $N$ such that every $k$-coloring of$\binom{[N]}{r}$ yield an $i$- homogeneous set of size $p_i$ for some i.

**[4] G. Suresh Singh, *Graph Theory*, 2010**

This book written by G. Suresh Singh published by PHI Learning Private Limited, New Delhi in 2010. This book also says about Ramsey theory. It give simple definition to Ramsey number and Ramsey graph.Given any two positive integer $k$ and $l$, there is a least positive integer denoted by $R(k, l)$ vertices contains either $K_k$ or $K_l^c$. The number $R(k, l)$ is called **Ramsey number**. He gives the property of Ramsey number. A $(k, l)$ **Ramsey graph** is a graph on $R(k, l)$-1 vertices that contains neither a $K_k$ nor $K_l^c$.

**[5] Lane Barton, *Ramsey Theory*, 2016**

It is a paper work of Lane Barton. He published his paper work in 2016, may 13. In his work he mentioned Paul Erdos Theorem and Van der Wardens Theorem.
The concept of Bounds of Ramsey Theory is also contributed by Erdos. In this paper we can see Van der warden's Theory. In this paper he proved, the properties of Ramsey Number.

**[6] *Ramsey Theory on Graphs*, 2020**

It is a mini project published by Department of Mathematics, The University of Manchester, on November 3 in 2020. This project includes some definitions and theorems related to Ramsey Theory. Theorems are similar to some other books.

# Chapter 3

# Definitions

In this chapter we will discuss some of the selected definition from Graph theory that are applicable in Ramsey theory.

## 3.1 Basic Definitions of Graph Theory

**Definition 3.1.1.** A graph is an ordered triple $G = (v(G), E(G), I(G))$ where $V(G)$ is nonempty set, $E(G)$ is a set disjoint from $V(G)$, and $I(G)$ is an incidence relation that associates with each element of $E(G)$ an unordered pair of elements of $V(G)$. Elements of $V(G)$ are called the vertices of $G$ and elements of $E(G)$ are called edges of $G$. $V(G)$ and $E(G)$ are vertex set and edge set of $G$, respectively. If for the edges $e$ of $G$, $I(G)()e= u, v$ we write $I(G)(e) = uv$

**Definition 3.1.2.** Let $G$ be a graph and $v \in V$. The number of edges incident at $v$ in $G$ is called the degree of vertex $v$ in $G$ and is denoted by $d(v)$. The minimum and maximum of the degrees of the vertices of a graph are denoted by

$\delta$ and $\Delta$ respectively.

**Definition 3.1.3.** A graph is called simple, it has no loop and parallel edge. Loop means a vertex $u$ joint to itself by an edge. Parallel edge means let $G$ be a graph if two or more edges of $G$ has same end vertices then these edges are called parallel.

**Definition 3.1.4.** A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge

**Definition 3.1.5.** If for some positive integer $K$, $d(v) = K$ for every vertex $v$ of graph $G$. $G$ is called $K$-regular

**Definition 3.1.6.** Let simple graph is called self compliment if it is isomorphic its on compliment

**Definition 3.1.7.** A graph $G$ is called connected if every two of its vertices are connected

A graph that is not connected is called disconnected

**Definition 3.1.8.** A graph $G$ is called a tree if it is a connected acyclic graph. Acyclic graph is a graph $G$ contain no cycle.

**Definition 3.1.9.** An independent set $S$ in a graph $G$ is a subset of vertices where for any two vertices $u$ and $v$ in $S$, there is no edge between $u$ and $v$ in the graph $G$.

**Definition 3.1.10.** A clique of $G$ is a complete subgraph of $G$. A clique of $G$ is a maximal clique of $G$ if it is not properly contained in another clique of $G$.

## 3.2 Definitions on Ramsey Theory

### 3.2.1 Pigeonhole Principle

The Pigeonhole principle, also known as the Dirichlet pigeonhole principle, simply states that if there exist $n$ pigeonholes containing $n+1$ pigeons, one of the pigeonholes must contain at least two pigeons . This can be generalised to say that if there are a finite number of pigeonholes must contain an infinite number of pigeons.

**Definition 3.2.1.** A 2-coloured graph is a graph is a graph whose edges have been coloured with 2 different colours.

### 3.2.2 Monochromatic Graph

**Definition 3.2.2.** Given an assignment of colours to all edges of a graph $G$, a subgraph $H$ of $G$ is called **monochromatic** if all the edges of $H$ have the same colour.

**Example 3.2.1.** Consider the complete graph $G = K_6$. Represent the six people by the six vertices of G. Any two vertices $a$ and $b$ are defined to be adjacent, if and only if the corresponding persons know each other. Now colour their edges by blue. If the person don't know each other, colour their edges between the corresponding vertices by red.

If there are three people who know each other then this is represented by blue triangle in $K_6$. Similarly, if there are three people who don't know each other then this is represented by a red triangle.

Take an arbitrary vertex a . Its degree is five in $K_6$. When we colour edges incident with a either blue or red. So one colour must be used at least three times. Assume that $ab, ac, ad$ are coloured blue. That means $a$ and $b$, $a$ and $c$, $a$ and $d$ knows each other. If $bc, bd$, or $dc$ coloured blue then we get blue triangle.

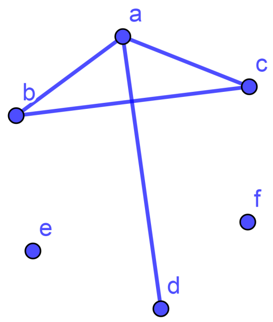


Figure 3.1: abc blue triangle

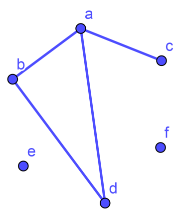


Figure 3.2: abd blue triangle

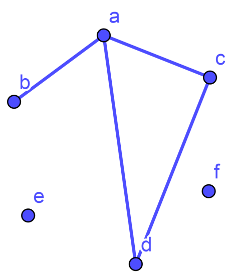


Figure 3.3: acd blue triangle

Similarly we can say in the case of red triangle. [1]

### 3.2.3 Ramsey Number

**Definition 3.2.3.** Given any two positive integers $k$ and $l$, there is a least positive integers denoted by $R(k,l)$ such that every graph on $R(k,l)$ vertices contains either $K_k$ or $K_l^c$. The numbers $R(k,l)$ is called **Ramsey numbers**.

We can use $r$ instead of $R$. We can defined in general terms. Let $r$ and $p_1, \cdots p_k$ be positive integers. If there is a $N$ such that every $k$-coloring of $\binom{[N]}{r}$yields an i-homogeneous set of size $p_i$ for some i, then the smallest such integer is called **Ramsey number** $R(p_1, \cdots p_k : r)$. Ramsey theorem states that we can find such integer for every choice of $r$ and $p_1, \cdots p_k$.

**Remark 1.** (1) $R(k,l) = 1 = R(1,l)$

(2) $R(2,l) = l, R(k,2) = k$

(3) Every graph on $p$ vertices contain either a clique of $k$ vertices or an independent set of l vertices if $p \geq R(k,l)$

(4) If there is a graph on $p$ vertices that contains neither a clique of $k$ vertices nor an independent set of vertices then
$p < R(k,l)$ or $p + 1 \leq R(k,l)$

(5) $G$ contains $K_k$ or $K_l^c$ if and only if $G^c$ contains $K_l^c$ or $K_l$.

**Example 3.2.2.** $R(3,3)$=6

Also it can ask like this: What is the smallest number of people so that there must be 3 mutual friend and 3 mutual strangers?

**Solution**: Let $v_1, v_2, v_3, v_4, v_5, v_6$ be 6 vertices. We will colour blue to represent friends and red to represent strangers. By Pigeonhole principle, "if n items are put into m containers, with n > m, then at least one container must contain more than one". So, we will colour 5 edge by two colour, that means at least 3 edges are coloured with same colour.

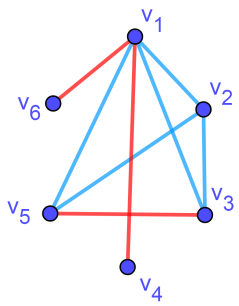


Figure 3.4: $K_3$ Graph

By this figure we get $K_3$ graph. Suppose $v_2$, $v_5$ coloured by red. And $v_3$, $v_2$ coloured by red

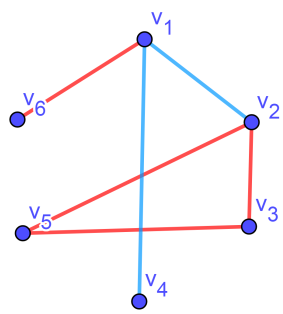


Figure 3.5: Independent of 3 vertices

We get independent set of 3 vertices. We can coloured by any way, we will get either $K_3$ or independent of 3 vertices.

Next we will show the smallest number of vertices in these type of graph is 6. Consider a graph with 5 vertices. We need to check $K_3$ or independent set of 3 vertices.

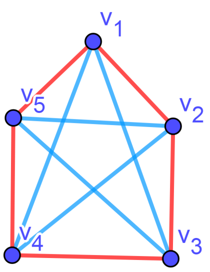


Figure 3.6: Graph $K_5$

There is neither clique of 3 vertices nor independent set of 3 vertices. Therefore, the smallest positive number of vertices to form 3 vertices or independent set of 3 vertices is 6

**Example 3.2.3.** Show $R(3,4)$=9?

**Solution**: We have, If $k \geq 2$ and $l > 2$, then

$$\begin{aligned} R(k,l) &\leq R(k-1,l) + R(k,l-1) \\ R(3,4) &\leq R(2,4) + R(3,3) \\ &\leq R(2,4) + R(3,3) - 1 \\ &< R(2,4) + R(3,3) \end{aligned}$$

$$< 4 + 6 = 10$$

$$\leq 9$$

We need to prove $R(3, 4) > 8$

Assume that not less than or equal to 8. Let $G$ be with 8 vertices. We need to check either there exist $K_3$ or independent of 4 vertices.

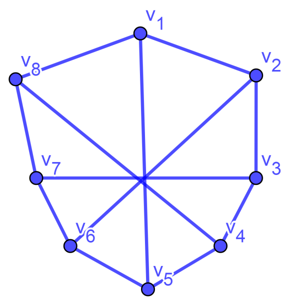


Figure 3.7: Graph with 8 vertices

This is the graph with 8 vertices. The vertices are connected if $i - j$ =1 or 4. In this graph we can see the connected vertices. Consider any 4 vertices.

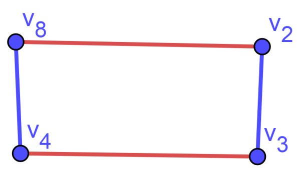


Figure 3.8: 4 vertices graph

We cannot find a $K_3$ or independent set. So our assumption is wrong. $R(3,4) < 8$. This implies $R(3,4) = 9$.

| s. t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | 1 | 3 | 6 | 9 | 14 | 18 | 23 | 28 | 36 | |
| 4 | 1 | 4 | 9 | 18 | 25 | | | | | |
| 5 | 1 | 5 | 14 | 25 | | | | | | |
| 6 | 1 | 6 | 18 | | | | | | | |
| 7 | 1 | 7 | 23 | | | | | | | |
| 8 | 1 | 8 | 28 | | | | | | | |
| 9 | 1 | 9 | 36 | | | | | | | |
| 10 | 1 | 10 | | | | | | | | |

Table 3.1: Ramsey Table

### 3.2.4 Ramsey Graph

**Definition 3.2.4.** A $(k,l)$ **Ramsey graph** is a graph on $R(k,l)$-1 vertices that contains neither a $K_k$ nor $K_l^c$.

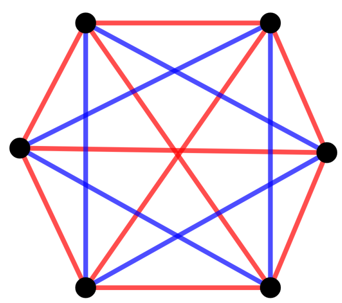

Figure 3.9: Ramsey graph

### 3.2.5 $k$-colouring of a set

**Definition 3.2.5.** A **$k$-partition or $k$ colouring** of a set is a partition of it into $p$ classes. A class or is label is a colour. Typically, the set of colours is $[p]$. The collection of $r$-sets in $S$ is $\binom{S}{r}$; a $k$-colouring of these is a map $f : \binom{S}{r} \to [k]$. A homogeneous set is a set $T \subseteq S$ whose $r$-sets have received the same colour; if colour $i$, then $T$ is **$i$-homogeneous**. The notion $N \to (k_1, k_2 \cdots, k_p)^r$ means that every $k$ colouring of$\binom{[N]}{r}$ yields for some $i$, and $i$-homogeneous set of size $k_i$. such an integer $N$ exists, then the smallest such integer is also Ramsey number denoted by $R(k_1, K_2, \cdots k_p : r)$. [3]

# Chapter 4

# General Results on Ramsey Theory

In this chapter we deal with theorems and proof, also results related on Ramsey Theory.

**Theorem 4.0.1.** *R(1, l)= R(k, 1) = 1*

*Proof.* $R(1, l)$ it means the graph contains either a clique of 1 vertex or independent set of $l$ vertices.
Let $G$ have only 1 vertex. It contain clique of 1 vertex..
Let $G'$ has 2 vertices that is $v_1, v_2$. If it is connected by an edge $e$. Then $\exists$ a clique of 1 vertex. Similarly in $n$ number of vertices. By the definition of Ramsey number, we get $R(1, l) = 1$
$R(k, 1)$ it means graph contain either a clique of $k$ vertices or independent set of 1 vertex
Let $G$ be a graph with 1 vertex. Then we get independent set of 1 vertex.
Let $G'$ be a graph with 2 vertices $v_1, v_2$, if there is a edge connected by an edge. So there exist clique of 2 vertices, otherwise independent set of 1 vertices.

Similarly, in the case of $3, 4, \ldots$ number of vertices.

By the definition of Ramsey number, we get $R(k, 1) = 1$.

$\implies R(1, l) = R(k, 1) = 1$ □

**Theorem 4.0.2.** *For any positive integer $k$ and $l$ $R(k, l)$ =$R(l, k)$*

*Proof.* Let $R(k, l) = p$. To prove the theorem, its enough to prove that $R(l, k) = p$.

Let $G$ be a graph with order $p$.

**Claim**: Either $K_l \subseteq G$ or $K_k \subseteq G^c$

Since $G$ is of order $p$, it is clear that $G^c$ is also of order $p$. Since $R(k, l) = p$, either $K_k \subseteq G^c$ or $K_l \subseteq (G^c)^c = G$, which implies either $K_l$ is a subgraph of $G$ or $K_k$ is a subgraph of $G^c$. Hence the claim. Thus,

$$R(l, k) \leq p \tag{4.1}$$

Further, since $R(K, l) = p$, there is a graph $H$ with order $p - 1$ such that $K_k$ is not a subgraph of $H$, and $K_l$ is not subgraph of $H^c$. It follows that $K_l$ is not a subgraph of $H^c$ of order $p - 1$ and $K_k$ is not a subgraph of $(H^c)^c = H$, which implies that

$$R(l, k) \geq p \tag{4.2}$$

From Eqs. (4.1) and (4.2), we get $R(l, k) = p$.

Hence the proof □

## 4.1 Existence of Ramsey Number

We have some values for Ramsey Numbers to find them. In this section we show that in fact the Ramsey Number exists for all $k, l \geq 2$, even if we don't know the value is. By this section we prove a method to find Ramsey Number. [3]

**Theorem 4.1.1.** *For any positive two integers $k \geq 2$ and $l \geq 2$ $R(k,l) \leq R(k,l-1) + R(k-1,l)$*
*Furthermore if $R(k,l-1)$ and $R(k-1,l)$ are both even, then inequality strictly hold.*

*Proof.* Let $R(k,l-1) + R(k-1,l) = n$. Let $G$ be graph with $n$ vertices. Let $v \in V(G)$.
Then there exit other $n-1$ vertices, which implies there exist other $R(k,l-1) + R(k-1,l) - 1$ vertices other than $v$. We will get two cases.
**Case 1**: $v$ is non adjacent to a set $S$ at least $R(k,l-1)$ vertices.
$V(S) \geq R(k,l-1)$.
**Case 2**: $v$ is adjacent to a set $T$ at least $R(k-1,l)$ vertices.
$V(T) \geq R(k-1,l)$.
Either of case 1 or case 2 must hold.
Suppose neither case 1 nor case 2 holds.
That is,

$$V(S) < R(k,l-1) \leq R(k,l-1) - 1$$
$$V(T) < R(k-1,l) \leq R(k-1,l) - 1.$$

We have, number of vertices to which $v$ is adjacent + number of vertices to which $v$ is non adjacent.

$$\begin{aligned} &\leq R(k,l-1)-1+R(k-1,l)-1 \\ &= R(k,l-1)+R(k-1,1)-2 \\ &< R(k,l-1)+R(k-1,l)-1 \end{aligned}$$

But LHS becomes $n-1$. we get $n-1<n-1$. This is a contradiction. So our assumption is wrong. Therefore, either 1 or 2 will hold.

Assume that case 1 hold. There form $G[S]$ a subgraph $S$ with at least $R(k,l-1)$ vertices. In $S$ $\exists$ clique of $k$ vertices or an independent set of $l-1$ vertices. Consider $G[S\cup\{v\}]$ contains a clique of $k$ vertices or an independent set of $l$ vertices. Assume that case 2 holds. Then $v$ is adjacent to set $T$ of at least $R(k-1,l)$ vertices. $G[T]$ forms subgraph $T$ with at least $R(k-1,l)$ vertices.

In $T$, $\exists$ a clique of $k-1$ vertices or an independent set of $l$ vertices. Consider $G[T\cup\{v\}]$ contains a clique of $k$ vertices.

In either case $\exists$ $R(k,l)$ set of vertices. We know that it is the least one. So, $R(k,l)\leq n$ Suppose both are even. Let $G$ be graph of $R(k,l-1)+R(k-1,l)$ -1 vertices. In any graph $G$ number of vertices of odd degree is even. This implies $\exists$ some vertex $v$ is even degree. That is incident to even number of vertices. Since $v$ has even degree $v$ cannot adjacent to precisely $R(k-1,l)-1$ vertices. Let $T$ be number of adjacent vertices to $v$ and $S$ be non adjacent vertices.

$|T|$ cannot same as $R(k-1,l)-1$. Either $(a)$ $|T|<R(k-1,l)-1$ or
$(b)$ $|T|>R(k-1,l)-1$.

$$|T|\neq R(k-1,l)-1$$

$$|S|\neq R(k,l-1)-1$$

$|T| \geq R(k-1, l)$, then $(b)$ satisfy. If $|S| \geq R(k, l-1)$, then $(a)$ satisfy. Thus $G$ contains clique of $k$ vertices or independent set of $l$ vertices.

Hence the proof. □

**Example 4.1.1.** We know $R(3,3) = 6$. We prove it by theorem.

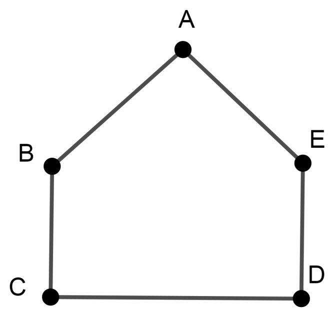


Figure 4.1: 5 vertices graph

This graph contains neither $K_3$ nor $K_3^c$. Thus,

$$R(3,3) > 5 \text{ nor } R(3,3) \geq 6$$

By theorem $R(3,3) \leq R(3,2) + R(2,3) = 3 + 3$. Thus,

$$R(3,3) \leq 6$$

. That is we get

$$R(3,3) = 6$$

**Example 4.1.2.** Consider $R(3,5)$

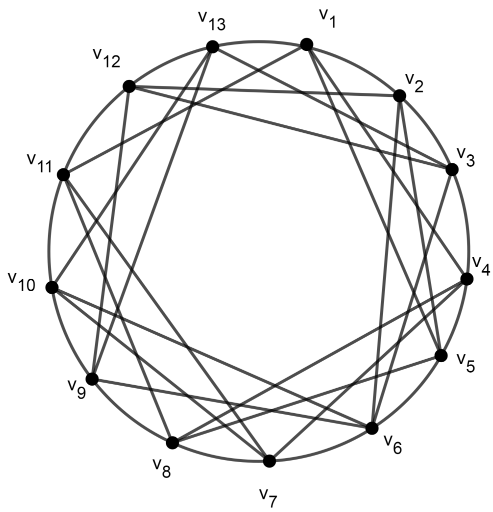


Figure 4.2

This graph neither contains a $K_3$ nor $K_3^c$. Thus,

$$R(3,5) > 13 \text{ or } R(3,5) \geq 14.$$

We have $R(3,5) \leq R(3,4) + R(2,5) = 9 + 5$.

$$R(3,5) \leq 14$$

Therefore,

$$R(3,5) = 14$$

## 4.2 Bounds of Ramsey Numbers

However, there are bounds that can allow us close to the actual value. Further reading on recent developments on upper and lower bounds on Ramsey Number.

In this section, we will see the proof for upper bound and lower bound

**Theorem 4.2.1.** *For any two positive integers $k, l$, $R(k,l) \leq \begin{pmatrix} k+l-2 \\ k-1 \end{pmatrix}$*

*Proof.* This proof by induction on $k+l$.

$R(1,1) = 1$. Here $k = 1$ and $l = 1$. By hypothesis, $k+l-2 = 0$ and $k-1 = 0$ $\binom{0}{0} = 1$.

Similarly, $R(1,2) = 1$, $R(2,2) = 2$, $R(2,3) = 3$.

This result is true for $k+l \leq 5$.

Assume that the result true for $5 \leq k+l < m+n$.

Next we will prove that the result is true for $m+n$. That is,

$$R(m,n) \leq \begin{pmatrix} m+n-2 \\ m-1 \end{pmatrix}$$

By previous theorem,

$$R(m,n) \leq R(m-1,n) + R(m,n-1)$$

we get,

$$\begin{aligned} R(m,n) &\leq \begin{pmatrix} m-1+n-2 \\ m-1-1 \end{pmatrix} + \begin{pmatrix} m+n-1-2 \\ m-1 \end{pmatrix} \\ &\leq \begin{pmatrix} m+n-3 \\ m-2 \end{pmatrix} + \begin{pmatrix} m+n-3 \\ m-1 \end{pmatrix} \\ &= \frac{(m+n-3)!}{(m-2)!(n-1)!} + \frac{(m+n-3)!}{(m-1)!(n-2)!} \end{aligned}$$

multiplied first term by $(m-1)!$ and and second term by $(n-1)!$. We get,

$$\begin{aligned}
R(m,n) &\leq \frac{(m+n-3)!(m-1)!}{(m-2)!(n-1)!} + \frac{(m+n-3)!(n-1)!}{(m-1)!(n-2)!} \\
&= \frac{(m+n-3)![m-1+n-1]}{(m-1)!(n-1)!} \\
&= \frac{(m+n-3)!(m+n-2)}{(m-1)!(n-1)!} \\
&= \frac{(m+n-2)!}{(n-1)!(m-1)!} \\
&= \binom{m+n-2}{m-1} \\
ie, R(m,n) &\leq \binom{m+n-2}{m-1}
\end{aligned}$$

So the result is true for $m+n$. Therefore, $R(k,l) \leq \binom{k+l-2}{k-1}$.

Hence the proof ☐

**Corollary 4.2.1.** *For all $k \geq 2$, $k$ be an integer. We have that $R(k,k) \leq 4^{k-1}$*

*Proof.* Put k = l, into previous theorem. we have

$$R(k,k) \leq \binom{2k-2}{k-1}.$$

Now recall that the total number of subsets of a set of size $n$ is $2^n$. By the definition of binomial coefficient we are choosing subsets of size $2k-2$.

$$R(k,k) \leq \binom{2k-2}{k-1} \leq 2^{2k-2} = 2^{2(k-1)} = 4^{k-1}$$

□

**Example 4.2.1.** Show that $R(4,3) \geq 9$.

*Proof.* To see that $R(4,3) \geq 9$, consider the following two-coloured $K_8$. Label the vertices of the graph clockwise with elements of $1, 2, \cdots, 8$ in ascending order. Let edge $ij \in E(K_8)$, with $i < j$ be blue if $i - j \in \{1, 4, 7\}$ and red if $i - j \in \{2, 3, 5, 6\}$.

Suppose there exists a blue triangle. Then take a smallest vertex, $i$, of this triangle. Vertex $i$ must make a blue triangle with two of the vertices $i+1, i+4$ and $i+7$ for these edges to be blue. However, observe that none of these three vertices have difference 1, 4 or 7, and so cannot have a blue edge between them, a contradiction. Hence there is no blue triangle.

On the other hand, suppose we have a red $K_4$. Take smallest vertex $i$ of this red $K_4$. We would then need three of the four vertices $i+2, i+3, i+5$ and $i+6$. Observe that we must choose $i+2$ and $i+3$ together, or $i+5$ and $i+6$ together. However, their difference is 1 and so would be joined with a blue edge, a contradiction. Hence there is no red $K_4$. □

**Theorem 4.2.2.** *For any positive integer $k$, $R(k,k) \geq 2^{k/2}$. [3]*

*Proof.* Since $R(1,1) = 1$ and $R(2,2) = 2$. We assume that the result is true for $k \geq 3$. We have to show that $R(k,k) \geq 2^{k/2}$. Enough to show that $\exists$ no number $k$ such that $R(k,k) < 2^{k/2}$. That implies $R(k,k) \not< 2^{k/2}$. That is, if we choose any number $n$ with $n < 2^{k/2}$. Then there is a graph on $n$ vertices which is not

$R(k,k)$. For any $n < 2^{k/2}$, there exist a graph on $n$ vertices which has neither a $k$-clique nor $k$- independent set of vertices. This proof by probablistic method. We have to prove the existence of such graph.

Let $k \geq 3$

Let $G_n$ is set of possible simple graph with vertex set $\{v_1, v_2, \cdots v_n\}$

Let $G_n^k$ be the set of those graph in $G_n$ that have a clique of $k$ vertices. Let $E$ be the maximum number of edges in the set $\{v_1, v_2, \cdots v_n\}$. Then $|E| = \binom{n}{2}$

$$|G_n| = P(E) = 2^{\binom{n}{2}} \tag{4.3}$$

Clearly we can say $G_n^k \subset G_n$. Fix $S \subseteq V$ with $|S| = k$. The possible number of edges between vertices of $S$ is $\binom{k}{2}$. The rest of edges is $\binom{n}{2} - \binom{k}{2}$. Let $E'$ be any subset of $\binom{n}{2} - \binom{k}{2}$ edges with $k$-clique

$$|E'| = 2^{\binom{n}{2} - \binom{k}{2}}$$

Since $S$ is arbitrary, there exists $\binom{n}{k}$ distinct $k$ element subset of $\{v_1, v_2, \cdots v_n\}$

$$|G_n^k| \leq \binom{n}{k} 2^{\binom{n}{2} - \binom{k}{2}} \tag{4.4}$$

Divide Eqn (4.4) by (4.3)

$$\begin{aligned} \left| \frac{G_n^k}{G_n} \right| &\leq \frac{\binom{n}{k} 2^{\binom{n}{2} - \binom{k}{2}}}{2^{\binom{n}{2}}} \\ &= \frac{n!}{k!(n-k)!} 2^{\binom{-k}{2}} \\ &< \frac{n^k}{k!} 2^{\binom{-k}{2}} \end{aligned}$$

That is,

$$\left|\frac{G_n^k}{G_n}\right| < \frac{n^k}{k!}2^{\binom{-k}{2}} \tag{4.5}$$

Suppose we have a large $n$ such that Eqn (4.5) implies

$$\begin{aligned}\frac{|G_n^k|}{|G_n|} &< \frac{2^{(\frac{k}{2})k}2^{\binom{-k}{2}}}{k!}\\ &= \frac{2^{k^2/2}2^{\frac{-k(k-1)}{2}}}{k!}\\ &= \frac{2^{k/2}}{k!}\\ &< 1/2\end{aligned}$$

That is,

$$\tfrac{|G_n^k|}{|G_n|} < 1/2$$

$$|G_n^k| < 1/2|G_n| \tag{4.6}$$

$G_n = \{G|G^c \in G_n\}$.

Let $c_{G_n^k}$ be collection of simple graph in $G_n$ with $k$ independent set of vertices.

We know, $|c_{G_n^k}| = |G_n^k|$

$$|c_{G_n^k}| < 1/2|G_n| \tag{4.7}$$

$$\begin{aligned}|c_{G_n^k}| + |G_n^k| &< 1/2|G_n| + 1/2|G_n|\\ &< |G_n|\end{aligned}$$

This implies, $c_{G_n^k} \cup G_n^k \subset G_n$.

That is , $[c_{G_n^k} \cup G_n^k]^c \neq \varnothing$, which implies $\exists$ some graph $G_n$ that contain neither a $k$ clique nor $k$ independent set of vertices.This hold because of $n < 2k/2$. So,

$R(k,k) \geq 2^{k/2}$.

Hence the proof. $\square$

# Chapter 5

# Application and Conclusion

In this chapter we discuss application of Ramsey Theory. Then we conclude our project.
There are many interesting application of Ramsey Theory, in number theory, algebra, geometry, topology, set theory, logic, ergodic theory, information theory and theoretical computer science.

## 5.1 Confusion channel for noisy channels

In communication theory, a noisy channel gives rise to a confusion graph, a graph whose vertices are elements of a transmission alphabet $T$ and which has an edge between two letters of $T$ if and only if, when sent over the channel, they can be received as the same letter. Given a noisy channel, we would Like to make errors impossible by choosing a set of signals that can be unambiguously received, that is, so that no signal in the set is confusable with another signal in the set. This corresponds to choosing an independent set in the confusion graph

$G$. In the confusion graph $G$ of Fig. 5.1 the largest independent set consists of two vertices. Thus, we may choose two such letters, say $a$ and $c$, and use these as an unambiguous code alphabet for sending messages. In general, the largest unambiguous code alphabet has $\alpha(G)$ elements, where $\alpha(G)$is the size of the largest independent set in $G$.

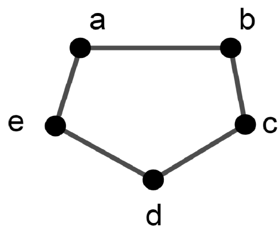


Figure 5.1: Confusion Graph 'G'

To see whether we can find a better unambiguous code alphabet, we shall introduce the notion of normal product $G.\ H$ of two graphs $G$ and $H$. This is defined as follows. The vertices are the pairs in the Cartesian product
$V(G) \times V(H)$. There is an edge between $(a, b)$ and $(c, d)$ if and only if one of the following holds:

(i) $\{a, c\} \in E(G)$ and $b, d \in E(H)$,

(ii) $a = c$ and $\{b, d\} \in E(H)$,

(iii) $b = d$ and $\{a, c\} \in E(G)$.

We can find a larger unambiguous code alphabet by allowing combinations of letters from the transmission alphabet to form the code alphabet. For example, suppose that we consider all possible ordered pairs of elements from the transmission alphabet $T$, or strings of two elements from $T$. Then under the confusion graph of Fig. 5.1, we can find four such ordered pairs, $aa$, $ac$, $ca$, and $cc$, none of which can be confused with any of the others. In general, two strings of letters from the transmission alphabet can be confused if and only if they can be received as the same string. In this sense, strings $aa$ and $ac$ cannot be confused, since $a$ and $c$ cannot be received as the same letter. We can draw a new confusion graph whose vertices are strings of length two from $T$. This graph has the following property: Strings $xy$ and $uv$ can be confused if and only if one of the following holds:

(i) $x$ and $u$ can be confused and $y$ and $v$ can be confused,

(ii) $x = u$ and $y$ and $u$ can be confused,

(iii) $y = u$ and $x$ and $u$ can be confused.

In terms of the original confusion graph $G$, the new confusion graph is the normal product $G \cdot G$. If $G$ is the confusion graph of Fig. 5.1, we have already observed that one independent set or unambiguous code alphabet in $G \cdot G$ can be found by using the strings $aa$, $ac$, $ca$, and $cc$. However, there is a larger independent set that consists of the strings $aa$, $bc$, $ce$, $db$, and $ed$.
The most famous theorem about it is Hedrlin theorem.

**Theorem 5.1.1.** *If $G$ and $H$ are any graphs, then*
$\alpha(G \cdot H) \leq R(\alpha(G) + l, \alpha(H) + l) - 1.$

Alon and Orlitzky use probabilistic constructions of self-complementary Ramsey graphs, that are also Cayley graphs. [6]

## 5.2 Design of packet switched networks

Stephanie Boyles and Geoff Exoo (personal communication) have found an application of Ramsey theory in the design of a packet switched network, the Bell System signaling network. We describe the application in this section.

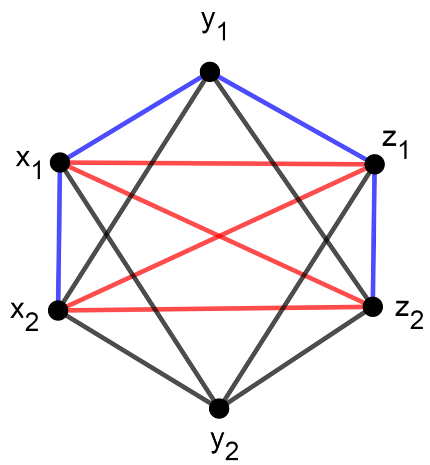


Figure 5.2: Link are coloured by red, blue, black

Consider a graph in which vertices represent communications equipment joined by communications links or edges. The graph is assumed to be complete, that is, every pair of vertices is joined by a link. In some applications, vertices are paired up, and we would like to guarantee that in case of outages of some links, there will always remain at least one link joining every paired set of vertices. For instance, consider the graph shown in Fig. 5.2. The vertices labeled $x_1$ and $x_2$ are paired, the vertices labeled $y_1$, and $y_2$ are paired, and the vertices labeled $z_1$, and $z_2$ are paired. Outages occur at intermediate facilities such as microwave towers, trunk groups, etc. An outage at such a facility will affect all links sharing this facility. Let us color the intermediate facilities and hence the corresponding links. Fig. 5.2 shows such a coloring. Note that in case the red intermediate facility goes out, there will be no operative links between the pair of vertices $x_1$, and $x_2$, and the pair of vertices $z_1$ and $z_2$. This corresponds to the fact that the four edges $\{x_i, z_j\}$ form a monochromatic (red) $Z_4$. In general, designing a network involves a decision as to the number of intermediate facilities and which links will use which intermediate facilities. We would like to design the network so that if any intermediate facility is destroyed, there will remain at least one link for each paired set of vertices. If the vertex pairing may change after the network is constructed, we want to avoid all monochromatic $Z_4$'s.

It turns out that $R(Z_4, Z_4, )$ =6. Thus, if there are just two intermediate facilities, there is a network with 5 vertices which has an assignment of links to intermediate facilities so that there is no monochromatic $Z_4$. Chung and Graham show that $R(Z_4, Z_4, Z_4) \geq 8$. Thus, there is a network with three different intermediate facilities and 7 vertices and no monochromatic $Z_4$.

As we have said, designing a network involves a decision as to the number of intermediate facilities and which links will use which intermediate facilities. Intermediate facilities are expensive, and it is desirable to minimize the number of them. Thus, one is led to ask the following. If we have a network of $n$ vertices, what is the least number of colors or intermediate facilities so that there is some network of $n$ vertices and some coloring of edges (assignment of links to intermediate facilities) with no monochromatic $Z_4$. In other words, what is the least $r$ so that if there are $rZ_4$'s, $R(Z_4, Z_4, \cdots Z_4) > n$? If $n = 6$, as in our example of Fig. 3, then since $R(Z_4, Z_4) = 6$, and $R(Z_4, Z_4, Z_4) \geq 8$, we have $r$= 3. We need three intermediate facilities. Boyles and Exoo point out that for their purposes, it is enough to estimate the number $r$ using a result of Erdos that a graph of $n$

vertices always contains $Z_4$, if it has at least $\frac{1}{2}n^{3/2}+\frac{1}{4}n$ edges. If the $\binom{n}{2}$ edges of an $n$-vertex graph are divided into $r$ color classes, the average class will have $\binom{n}{2}/r$ edges, and so, by the pigeonhole principle, some class will have at least $\binom{n}{2}/r$ edges. We want to be sure that no class has $\frac{1}{2}n^{3/2}+\frac{1}{4}n$ edges, so we must pick $r$ so that [6]

$$\binom{n}{2}/r > \frac{1}{2}n^{3/2}+\frac{1}{4}n$$

# 5.3 The Ramsey Pricing Theory

## 5.3.1 The Ramsey Pricing Method

Mathematical model of Ramsey pricing is expressed as follows. Suppose a public institution producing $n$ types of products (or provide $n$ types of customers with service), $i=1,2,\cdots,n$ , and the demand for all kinds of products (or groups of users) are independent. Product consumption is $q=(q_1,\cdots,q_n)$, when the price vector is $p=(p_1,\cdots,p_n)$, the demand function is $q_i=D_i(q_1,\cdots,q_n)$.

Enterprises Revenues: $R(q)=\sum_{i=1}^{n}p_iq_i$,

Cost Function: $C(q_1,\cdots q_n)$,

Total Consumer Surplus: $S(q)=\int D_i(q_i,\cdots,q_n)\cdot dq-\sum_{i=1}^{n}p_iq_i$,

Producer Surplus: $R(q)-C(q_1,\cdots q_n)$,

The Ramsey pricing problem is denoted as max.$\{S(q)+R(q)-C(q_1,\cdots q_n)\}$, such that $R(q)-C(q_1,\cdots q_n)\geq 0$..

The introduction of the Lagrange multiplier $\lambda$ makes the above maximization problem into:

$$W = \int D_i(q_1, \cdots, q_n) \cdot dq - C(q_1, \cdots q_n) + \lambda[R(q) - C(q_1, \cdots q_n)]$$

The formula of $q_i$ were seeking a first-order partial derivatives,

$$\begin{aligned}\frac{\partial W}{\partial q_i} = (p_i - MC_i) + \lambda\left(\frac{\partial p_i}{\partial q_i} q_i + p_i - MC_i\right) = 0 \\ \Longrightarrow (1+\lambda)(p_i - MC_i) = -\lambda \frac{\partial p_i}{\partial q_i} q_i \\ \Longrightarrow \frac{p_i - MC_i}{p_i} = \frac{\lambda}{1+\lambda} \cdot \frac{1}{n_i}\end{aligned}$$

Among them,

$$\eta_i = \frac{\partial q_i}{\partial p_i} \cdot \frac{p_i}{q_i}$$

is the price elasticity of demand of class I products (or class I user), if

$$R = \frac{\lambda}{1+\lambda}$$

is denoted as the Ramsey index, then

$$p_i = MC_i/(1 - R/\eta_i)$$

The Equation shows that the price of each type of product (user) on the marginal cost of an addition $(p_i - MC_i)/p_i$ is inverse proportion to the price elasticity of the product demand, this is the famous rally rules. Ramsey pricing consider the impact of user demand price elasticity to product prices, greater elasticity of demand for the product, if the price is too high, it will cause a significant decrease in demand, thus resulting in the decrease of consumer surplus and the loss of social welfare. If the price elasticity of demand for products with a lower price, due to less demand for change, the impact on consumer surplus and social welfare is relatively small.

### 5.3.2 The Analysis of the Ramsey Pricing Method

In order to better solve the problem of sales price cross-subsidies, we can apply the Ramsey pricing theory to determine the level of sales price of classified users. Appropriate adjustments to the existing sales price level with reference to Ramsey pricing, to reduce or cancel the purpose of cross subsidies.

Ramsey pricing is a sub optimal pricing method which is determined by the marginal cost of the enterprise and the price elasticity of different kinds of users in order to ensure the balance of power grid enterprises. The power cost of the residents is higher than that of the industrial users, and the price elasticity of the residents demand is less than that of the industrial users. In accordance with the Ramsey pricing theory, the sales price of the residents should be higher than the industrial customers, which is in line with the general situation of foreign residents in the electricity price is higher than the price of electricity .

The main steps of Ramsey pricing are shown in Figure 5.3. The difficulty lies in the determination of the marginal cost of power supply $MC_i$ and the price elasticity of demand for various types of users.

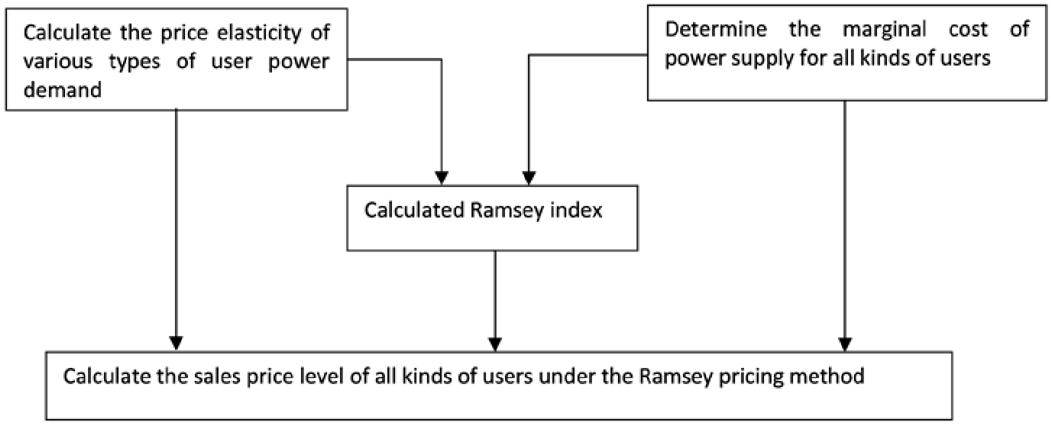


Figure 5.3: Outline of Ramsey Pricing Method

Under normal circumstances, the price elasticity of electricity demand increases with the increase of user power consumption. In accordance with the Ramsey pricing theory to determine the user's electricity price, industrial users demand price elasticity is higher and its power supply marginal cost is low, so the Ramsey tariff for industrial users should be low; in contrast, the demand price elasticity of residential users is small and its power supply marginal cost is higher, so it should be higher than the price of industrial electricity. To a certain extent, Ramsey price reduces the cross-subsidies in the past, and ensure the balance of corporate earnings under the premise to achieve the suboptimal goal of increasing social welfare, but it is not satisfactory in the social welfare distribution.

At present, China's current residential electricity price is lower than industrial tariffs. If you take Ramsey pricing method to determine the sales price of users, residential electricity prices will rise substantially, which is obviously unfair. The power of human life is an indispensable necessity, if substantial increase in residential electricity prices will lead to poor areas and low-income users not affording the high electricity bills, the impact of basic electricity needs of these users daily.

To solve the above problem, consider the lifeline tariff for residential users combined with Ramsey pricing, determine the lifeline of residential users per capita electricity consumption standards, within this standard residential electricity demand have greater rigidity. When residents of the user is within this standard, according to the lifeline electricity tariff to be settled; if residents lifeline electricity consumption exceeds the standard, the standard price of electricity by lifeline settlement, the excess electricity tariff by Ramsey settlement. This approach not only ensures the poor areas, low-income residents basic daily demand for electricity, but also encourages users to conserve electricity, to achieve optimal allocation of resources.

In the total welfare function, different users can set different welfare weights, and their mathematical expression is as follow:

$$(p_i - MC_i)/p_i = \lambda/\eta_i(\alpha_i = \lambda)$$

is social welfare weight factor. Government related departments can set or adjust the value $\alpha_i$ of residents, industry and other users, so as to achieve a reasonable adjustment of different kinds of user sales price, and to meet the purpose of different types of users of electricity demand.

# Conclusion

This project help us to understand Ramsey Theory in a detailed way.We initiated the project work by introducing of Ramsey Theory in which we discussed about invention and the reason for implement Ramsey Theory by Frank Plumpton Ramsey. Then we saw Ramsey number along with some examples and then we proved Ramsey theory. By application we understood that its development in various field. The complexity of Ramsey numbers and their applications in various fields emphasizes the presence of order over randomness.

Considering the implications of his findings, it turns out that Ramsey's theory goes beyond graph theory and affects the structure of mathematical structures. The project not only deepened our appreciation for the beauty of mathematical reasoning, but also provided valuable insights into the limitations and possibilities of the search for order.

In the great tapestry of mathematics, Ramsey's theory testifies to the inexhaustible richness of mathematical structures and their ability to inspire respect and curiosity. May our efforts contribute to ongoing dialogue in the field and ignite the intellectual curiosity of future generations of mathematicians.

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# Category Theory

Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics

by

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**Register No.CCAWMMS011**

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This is to certify that the project entitled **"CATEGORY THEORY "** submitted to Post Graduate and Research Department of Mathematics in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics, is a bonafide record of the project work done by **Ms.HIZANA V H (CCAWMMS011)** during the period of her study in the Post Graduate and Research Department of Mathematics, Christ College (Autonomous), Irinjalakuda, under my supervision and guidance during the year 2023-2024.

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First, there are no words to adequately acknowledge the wonderful grace that my Redeemer has given me. There are many individuals who have come together to make this project a reality. I greatly appreciate the inspiration; support and guidance of all those people who have been instrumental for making this project a success.

I express my deepest thanks to my guide Dr. Seena V, Assistant Professor, Department of Mathematics, Christ College(Autonomous), Irinjalakuda, who guided me faithfully through this entire project. I have learned so much from her, both in the subject and otherwise. Without her advice, support and guidance, it find difficult to complete this work.

I take this opportunity to express my thanks to our beloved principal Fr. Dr. Jolly Andrews CMI, who gave me the golden opportunity to do this wonderful project on the topic **"Category Theory"** .

I mark my sincere gratitude to Dr.Seena V, Head of the Department and to all other teachers of the department for providing me the necessary facilities to complete this project on time.

My project guide Dr. Seena V, deserves a special word of thanks for her invaluable and generous help in preparing this project in $LAT_EX$.

I want to especially thank all the faculty of the library for providing various

facilities for this project.

Words cannot express the love and support I have received from my parents, whose encouragement has buoyed me up from the beginning till the end of this work.

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# Contents

# Fuzzy Graph Theory

Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics

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# ACKNOWLEDGEMENT

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My heartfelt appreciation goes to all the faculty of the library for providing various facilities for this project.

I would also like to thank my parents for their constant encouragement.There are no words to describe the love and support I have gotten from my family whose inspiration has kept me going from the beginning till the end of this work.

**Jasna Varghese**

# Contents

# List of Figures

# Introduction

Graph Theory is the study of the relationships providing a helpful tool to quantify and simplify the moving parts of a dynamic system. Graph theory may be precisely traced back to 1735, when the Königsberg bridge issue was solved by Swiss mathematician Leonhard Euler. The Königsberg Bridge Problem is a well-known conundrum that involves figuring out a way to cross each of the seven bridges that span a forked river flowing past an island without going over any of them twice. Euler maintained that there isn't a way like that. He effectively proved the first theorem in graph theory with his demonstration, which simply made reference to the bridges physical configuration. Over a century after Euler's Königsberg bridge work, Cayley was motivated to investigate a specific class of graphs called trees because he was interested in certain analytical forms that emerged from differential calculus. His methods mostly involve counting graphs that possess specific attributes. Then the essential results published by Pólya between 1935 and 1937 and the results of Cayley led to the development of enumerative graph theory.

Many real life problems can be represented using Graph theory. But graphs do not represent all the systems properly due to the uncertainty or haziness of parameter of the system. This led to the introduction to fuzzy graphs. Rosenfeld first introduced the concept of fuzzy graphs. After that fuzzy graph thoery become a vast research area. Since the world is full of uncertainty, the fuzzy graphs occur in many real life situations. Fuzzy graph theory is advanced with large number of branches. In this project, the structural properties and some graph theoretical concepts of fuzzy graphs have been studied.

This work entitled **Fuzzy Graph Theory** introduces the concept of fuzzy-graphs and its various properties.

## Outline of the Project

Apart from the introductory chapter, we have described our work in four chapters.

**Chapter 1** covers the necessary definitions and concepts in basic fuzzy set theory and graph theory.

In **Chapter 2**, basic definitions, different types of fuzzy graphs and the connectedness of fuzzy graphs have been discussed.

In **Chapter 3**, the different types of operations on fuzzy graphs and associated matrices have been discussed.

In **Chapter 4** we look into the applications of fuzzy graph in various fields.

# Tensor Analysis

Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics

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# Contents

# Inverse Linear Programming

**Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics**

by


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**JOSNA J MANJALY**

# Contents

# List of Figures

# Introduction

The mathematical programming refers to mathematical models used to solve problems such as decision problems. The terms are meant to contrast with computer programming which solves such problems by implementing algorithms which may be designed specifically for a given problems.By mathematical programming,we consider declarative approaches.This means that a separation is considered between the representation of the problem through a mathematical model and its solving.The idea is that solving may be done through general methods such as branching methods,using the mathematical model designed to capture the problem.For instance,considering a decision problem that can be represented by a graph,variables can represent the presence or absence of such vertices and edges in the solution.

Linear programming or linear optimisation may be defined as the problem of maximising or minimising a linear function that is subjected to linear constraints. The constraints may be equalities or inequalities.The optimization problems. involve the calculation of profit and loss.Linear programming problems are the important class of optimization problems,that helps to find the feasible region and optimising the solution in order to give the highest or lowest value of the function.The main aim of the linear programming problem is to find the optimal solution.Linear programming is the method of considering different inequalities relevant to a situations and calculating the best value that is required to be obtained in those conditions.

In an optimisation problem, all parameters of the model are given,and we need to find from among all feasible solutions and optimal solution for a specified objective function.In an inverse optimisation problem,however,the situation is revised and we need to adjust the values of the parameters in a model has little as possible such that a given feasible solution becomes an optimal solution under the new parameter values.Sometimes the adjustment of various parameters cause different costs and the objective is to use a minimum cost to change the given feasible solution into an optimal one.This type of problems has an potential applications.

# Outline of the Project

Apart from the introductory chapter, we have described our work in four chapters.

**Chapter 1** In this chapter, we are focus on optimality conditions for this problem. we show that , under mild assumptions, these conditions can be checked in polynomial time.

In **Chapter 2**, we discuss a further study on inverse linear programming problems, which requires us to adjust the cost coefficients of a given problem.

In **Chapter 3**, The inverse optimal problem have been discussed.

In **Chapter 4**, We are focus on some application to inverse network optimization. The inverse LP problem can be used to analyse some inverse network optimization.

# Chapter 1

# Inverse Programming

## 1.1 Introduction

Let $\psi(b,c) = argmax\{C^T x : Ax = b, x \geq 0\}$ denote the set of optimal solutions of a linear parametric optimization problem

$$max\{c^T x : Ax = b, x \geq 0\} \qquad (1)$$

where the parameters of the right hand side and in the objective function are elements of given sets

$$\mathcal{B} = \{b : Bb = \tilde{b}\}, \mathcal{C} = \{c : Cc = \tilde{c}\}$$

respectively. Throughout this project, $A \in R^{m\times n}$is a matrix of full row rank $m, B \in R^{p\times m}$, $C \in R^{q\times n}$, $\tilde{b} \in R$ , $\tilde{c} \in R^q$. This data is fixed once and for all. Let$x^0 \in R^n$ also be fixed. Our task is to find values $\bar{b}$ and $\bar{c}$ for the parameters, such that $x^0 \in \psi(\bar{b},\bar{c})$ or, if this is not possible, $x^0$ is at least close to$\Psi(\bar{b},\bar{c})$. Thus we consider the following bilevel programming problem

$$\min \{\|x - x^0\| : x \in (b, c), b \in B, c \in C\}, \quad (2)$$

which has a convex objective function $x \in R^n \to f(x) := \|x - x^0\|$, but not necessarily a convex feasible region.

Throughout the project the following system is supposed to be infeasible:

$$A^T y = c$$

$$Cc = \tilde{c} \ . \qquad (3)$$

Otherwise every solution of

$$\begin{aligned} Ax &= b \\ x &\geq 0, \\ Bb &= \tilde{b}, \end{aligned} \tag{1.1}$$

would be feasible for (2), which means that (2) reduces to

$$min\{\|x - x^0\| : Ax = b, x \geq 0, Bb = \tilde{b}\},$$

which is a convex optimization problem.

## 1.2 Reformulation as an Mathematical program with equilibrium constraints

First we transform (2) via the Karush-Kuhn-Tucker conditions into a mathematical program with equilibrium constraints (MPEC) and we get

$$\begin{aligned}
\|x - x^0\| \rightarrow \; & min_{x,b,c,y} \\
Ax &= b \\
x &\geq 0 \\
A^T y &\geq c \\
x^T(A^T y - c) &= 0 \\
Bb &= \tilde{b} \\
Cc &= \tilde{c}.
\end{aligned} \tag{1.2}$$

The next thing which should be clarified is the notion of a local optimal solution.

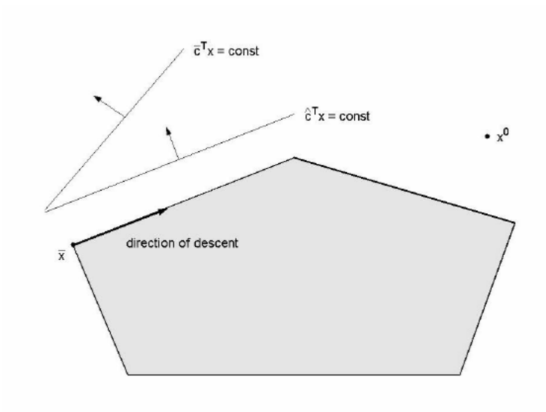


Figure 1.1: Definition of a local optimal solution

**Definition 1.2.1.** A point $\bar{x}$ is a local optimal solution of problem (2) if there exists a neighborhood $U$ of $\bar{x}$ such that $\|x - x^0\| \geq \|\bar{x} - x^0\|$ for all $x, b, c$ with $b \in \mathcal{B}$, $c \in C$ and $x \in U \cap \psi(b, c)$.

**Theorem 1.2.1.** Let $\mathcal{B} = \{\bar{b}\}$, $\{\bar{x}\} = \psi(\bar{b}, c)$ for all $c \in U \cap C$, where $U$ is some neighborhood of $\bar{c}$. Then, $(\bar{x}, \bar{b}, \bar{c}, \bar{y})$ is a local optimal solution of (1.2) for some dual variables $\bar{y}$.

## 1.3 Optimality via Tangent Cones

Now we consider a feasible point $\bar{x}$ of problem (2) and we want to decide whether $\bar{x}$ is local optimal or not. To formulate suitable optimality conditions certain subsets of the index set of active inequalities in the lower level problem need to be determined. Let

$$I(\bar{x}) = \{i : \bar{x}_i = 0\}$$

be the index set of active indices. Then every feasible solution $x$ of (2) close enough to $\bar{x}$ satisfies $x_i > 0$ for all $i \notin I(\bar{x})$. Complementarity slackness motivates us to define the following index sets, too:

- $I(c, y) = \{i : (A^T y - c)_i > 0\}$
- $\mathcal{I}(\bar{x}) = \{I(c, y) : A^T y \geq c, (A^T y - c)_i = 0 \forall i \notin I(\bar{x}), Cc = \tilde{c}\}$
- $I^0(\bar{x}) = \cap_{I \in \mathcal{I}(\bar{x})} I$.

**Remark 1.3.1.** If an index set $I$ belongs to the family $\mathcal{I}(\bar{x})$ then $I^0(\bar{x}) \subseteq I \subseteq I(\bar{x})$.

**Remark 1.3.2.** We have $j \in I(\bar{x}) \setminus I^0(\bar{x})$ if and only if the system

$$\begin{aligned} (A^T y - c)_i = 0 \quad \forall i \notin I(\bar{x}) \\ (A^T y - c)_j = 0 \\ (A^T y - c)_i \geq 0 \quad \forall i \in I(\bar{x}) \setminus \{j\} \\ Cc = \tilde{c} \end{aligned}$$

is feasible. Furthermore $I^0(\bar{x})$ is an element of $\mathcal{I}(\bar{x})$ if and only if the system

$$(A^T y - c)_i = 0 \quad \forall i \notin I^0(\bar{x})$$
$$(A^T y - c)_i \geq 0 \quad \forall i \in I^0(\bar{x})$$
$$Cc = \tilde{c}$$

is feasible.

**Lemma 1.3.1.** $\bar{x}$ is a local optimal solution of (2) if and only if $\bar{x}$ is a (global) optimal solution of all problems (1.3)

$$\begin{aligned} \|x - x^0\| &\rightarrow min_{x,b} \\ Ax &= b \\ x &\geq 0 \\ x_i = 0 \ \ \forall i &\in I \\ Bb &= \tilde{b} \end{aligned} \tag{1.3}$$

with $I \in \mathcal{I}(\bar{x})$.

*Proof.* Let $\bar{x}$ be a local optimal solution of (2) and assume that there is a set $I \in \mathcal{I}(\bar{x})$ with $\bar{x}$ being not optimal for (1.3). Then there exists a sequence $\{x^k\}_{k \in N}$ of feasible solutions of (1.3) with $lim_{k \rightarrow \infty} x^k = \bar{x}$ and $\|x^k - x^0\| < \|\bar{x} - x^0\|$ for all $k$.
Consequently $\bar{x}$ can not be a local optimal solution to (2) since $I \in \mathcal{I}(\bar{x})$ implies that all $x^k$ are also feasible for (2).

Conversely, let $\bar{x}$ be an optimal solution of all problems (1.3) and assume that there is a sequence $\{x^k\}_{k\in N}$ of feasible points of (2) with $lim_{k\to\infty}x^k = \bar{x}$ and $\|x^k - x^0\| < \|\bar{x} - x^0\|$ for all $k$. For $k$ sufficiently large the elements of this sequence satisfy the condition $x_i^k > 0$ for all $i \notin I(\bar{x})$ and due to the feasibility of $x^k$ for (2) there are sets $I \in \mathcal{I}(\bar{x})$ such that $x^k$ is feasible for problem (1.3) . Because $\mathcal{I}(\bar{x})$ consists only of a finite number of sets, there is a subsequence $\{x^{k_j}\}_{j\in N}$ where $x^{k_j}$ are all feasible for a fixed problem (1.3) .So we contradict the optimality of $\bar{x}$ for this problem (1.3).

□

**Corollary 1.3.1.** We can also consider

$$\begin{aligned} \|x - x^0\| &\to min_{x,b,I} \\ Ax &= b \\ x &\geq 0 \\ x_i = 0 \quad \forall i &\in I \\ Bb &= \tilde{b} \\ I &\in \mathcal{I}(\bar{x}) \end{aligned} \tag{1.4}$$

to check if $\bar{x}$ is a local optimal solution of (2). Here the index set $I$ is a minimization variable. This Problem (1.4) combines all the problems (1.3) into one problem and means that we have to find a best one between all the optimal solutions of the problems (1.3) for $I \in \mathcal{I}(\bar{x})$.

In what follow we use the notation

$T_I(\bar{x}) = \{d/\exists r : Ad = r, Br = 0, d_i \geq 0 \ \ \forall i \in I(\bar{x}) \setminus I, d_i = 0 \quad \forall i \in I\}$ .

This set corresponds to the tangent cone (relative to x only) to the feasible set of problem (1.3) at the point $\bar{x}$. The last lemma obviously implies the following necessary and sufficient optimality condition.

**Lemma 1.3.2.** $\bar{x}$ is a local optimal solution of (1.4) if and only if $f'(\bar{x}, d) \geq 0$ for all

$$d \in T(\bar{x}) := \cup_{I \in \mathcal{I}(\bar{x})} T_I(\bar{x}) \ .$$

**Remark 1.3.3.** $T(\bar{x})$ is the (not necessarily convex) tangent cone (relative x) of problem (1.4) at the point $\bar{x}$.

**Corollary 1.3.2.** The condition $I^0(\bar{x}) \in \mathcal{I}(\bar{x})$ implies $T_{I^0(\bar{x})}(\bar{x}) = T(\bar{x})$.

**Remark 1.3.4.** If f is differentiable at $\bar{x}$, then saying that $f'(\bar{x}, .)$ is nonegative over $T(\bar{x})$ is obviously equivalent to saying that

$$f'(\bar{x}, d) \geq 0 \ \ \forall d \in convT(\bar{x}), \tag{1.5}$$

where the "conv" indicates the convex hull operator.

## 1.4 A Formula for the tangent cone

For the verification of the optimality condition (1.5) an explicit formula for the tangent cone conv $T(\bar{x})$ is essential. For notational simplicity we suppose $I(\bar{x})$ $= \{1, ..., k\}$ and $I^0(\bar{x}) = \{l + 1, ..., k\}$ with $l \leq k \leq n$. Consequently all feasible points of (2) sufficiently close to $\bar{x}$ satisfy $x_i = 0$ for all $i \in I^0(\bar{x})$.

We pay attention to this fact and consider the following relaxed problem

$$\begin{aligned} \|x - x^0\| &\to min_{x,b} \\ Ax &= b \\ xi \geq 0, i &= 1, ..., l \\ x_i = 0, i &= l+1, ..., k \\ Bb &= \tilde{b}. \end{aligned} \tag{1.6}$$

In what follow we use the notation

$T_R(\bar{x}) = \{d|\exists r : Ad = r, Br = 0, d_i \geq 0, i = 1, ..., l, d_i = 0, i = l+1, ..., k\}$.

This set corresponds to the tangent cone (relative x) of (1.6) at the point $\bar{x}$. Since $I^0 \subseteq I$ for all $I \in \mathcal{I}(\bar{x})$, it follows immediately that

$$convT(\bar{x}) = coneT(\bar{x}) \subseteq T_R(\bar{x}). \tag{1.7}$$

The point $\bar{x}$ is said to satisfy the full rank condition, if

$$span(\{A_i : i \notin I(\bar{x})\}) = R^m, \tag{1.8}$$

where $A_i$ denotes the ith column of the matrix A

For Example;

All non-degenerate vertices of $Ax = b$, $x \geq 0$ satisfy (1.8).

# Chapter 2

# A further study on inverse linear programming problems

## 2.1 Inverse LP problem

Given a linear program

$$
\begin{aligned}
(LP)\text{Min} \quad & c^T x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

where $A \in R^{m\times n}$,$b \in R^m$ and $c, x \in R^n$, and let $x^0$ be a feasible solution, we consider the problem of changing the cost vector $c$ as less as possible such that $x^0$ becomes an optimal solution of (LP) under the new cost vector $\tilde{c}$. This inverse problem can be formulated as

$$\begin{aligned}(ILP)\text{Min} \quad & \|\tilde{c} - c\| \\ \text{s.t.} \quad & \pi p_j \leq \tilde{c}_j, j \in \underline{\text{J}}, \\ & \pi p_j = \tilde{C}_j, j \in J\end{aligned}$$

where $\underline{\text{J}} = \{j | x_j^0 = 0\}$, $\text{J} = \{j | x_j^0 > 0\}$, $P_j$ is the $j$-th column of A, $\pi$ is a row vector of dimension $m$, and $\|.\|$ is a vector norm.

Let $\tilde{c_j} = c_j + \theta_j$ -$\alpha_j$, and $\theta_j$, $\alpha_j \geq 0$ for $j = 1.....n$, where $\theta_j$ and $\alpha_j$ are respectively the increment and decrement of $c_j$. Notice that in our model $\theta_j \alpha_j = 0$, i.e. $\theta_j$ and $\alpha_j$ can never be positive at the same time. Then problem (ILP) can be expressed as

$$\begin{aligned}\text{Min} \quad & \|\theta + \alpha\| \\ \text{s.t.} \quad & \pi p_j - \theta_j + \alpha_j \leq c_j, j \in \underline{\text{J}}, \\ & \pi p_j - \theta_j + \alpha_j = c_j, j \in J \\ & \theta_j, \alpha_j \geq 0, j = 1, 2, ..., n\end{aligned} \tag{2.1}$$

Apparently, problem (2.1) is equivalent to

$$\begin{aligned}\text{Min} \quad & \|\theta + \alpha\| \\ \text{s.t.} \quad & \pi p_j - \theta_j \leq c_j, j \in \underline{\text{J}}, \\ & \pi p_j - \theta_j + \alpha_j = c_j, j \in J, \\ & \theta_j \geq 0, j \in \underline{\text{J}} \cup J, \\ & \alpha_j \geq 0, j \in J.\end{aligned} \tag{2.2}$$

Note that the second group of constraints in Eq. (2.2) can be expressed as

$$\begin{aligned}-\pi p_j + \theta_j - \alpha_j \geq -c_j, \\ \pi p_j - \theta_j + \alpha_j \geq c_j,\end{aligned} \tag{2.3}$$

which, under the condition $\theta_j, \alpha_j \geq 0$, imply

$$\begin{aligned} -\pi p_j + \theta_j &\geq -c_j, \\ \pi p_j + \alpha_j &\geq c_j. \end{aligned} \tag{2.4}$$

On the contrary, if Eq. (2.4) holds, and if $\theta_j > 0$ then in the optimal solution $\alpha_j = 0$ and $-\pi p_j + \theta_j = -c_j$,
which ensure the condition (2.3). If $\alpha_j > 0$, we again can derive Eq. (2.3). Therefore, problem (2.2) is equivalent to

$$\begin{aligned} \text{Min} \quad & \|\theta + \alpha\| \\ \text{s.t.} \quad & -\pi p_j + \theta_j \geq -c_j, j \in \underline{\mathrm{J}} \\ & -\pi p_j + \theta_j \geq -c_j, j \in J \\ & \pi p_j + \alpha_j \geq c_j, j \in J, \\ & \theta_j \geq 0, j \in \underline{\mathrm{J}} \cup J, \\ & \alpha_j \geq 0, j \in J. \end{aligned} \tag{2.5}$$

If the LP problem with bounded variables
(BLP)

$$\begin{aligned} \text{Min} \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & 0 \leq x \leq u \end{aligned}$$

is concerned, where $u$ is a given nonnegative vector, then for a given feasible solution $x^0$, the inverse problem can be formulated in a more symmetric form:

(IBLP)

$$\begin{array}{ll} \text{Min} & \|\theta + \alpha\| \\ \text{s.t.} & -\pi p_j + \theta_j \geq -c_j, j \in \underline{\text{J}} \cup J, \\ & \pi p_j + \alpha_j \geq c_j, j \in J \cup \bar{J}, \\ & \theta_j \geq 0, j \in \underline{\text{J}} \cup J, \\ & \alpha_j \geq 0, j \in J \cup \bar{J}, \end{array}$$

where

$\underline{\text{J}} = \{j|x_j^0 = 0\}$, $J = \{j|0 < x_j^0 < u_j\}$, and $\bar{J} = \{j|x_j^0 = u_j\}$.

## 2.2 The solution of (ILP) under $l_1$ norm

Under the $l_1$ norm, problem (2.5) becomes

(ILP1)

$$\begin{array}{ll} \text{Min} & \sum_{j=1}^n \theta_j + \sum_{j \in J} \alpha_j \\ \text{s.t.} & -\pi p_j + \theta_j \geq -c_j, j \in \underline{\text{J}} \cup J, \\ & \pi P_j + \alpha_j \geq c_j, j \in J, \\ & \theta_j \geq 0, j \in \underline{\text{J}} \cup J, \\ & \alpha_j \geq 0, j \in J, \end{array}$$

which is a LP problem with the dual

$$\begin{array}{ll} \text{Max} & -c^T x + c_J^T y \\ \text{s.t.} & Ax - A_J y = 0, \\ & 0 \leq x_j \leq 1, j \in \underline{\text{J}} \cup J, \\ & 0 \leq y_j \leq 1, j \in J, \end{array}$$

where $A_j$ is the submatrix of $A$ consisting of the columns $p_j$ for $j \in J$, and $c_J$ is the subvector of $c$ consisting of the components $c_j$ for $j \in J$. If we let

$$z_j = \begin{cases} x_j, \quad j \in \underline{\mathrm{J}}, \\ x_j - y_j, \quad j \in J, \end{cases}$$

then the above dual problem can be rewritten as

(DILP1)

$$\begin{array}{ll} \text{Max} & -c^T z \\ \text{s.t.} & Az = 0, \\ & 0 \leq z_j \leq 1, j \in \underline{\mathrm{J}}, \\ & -1 \leq z_j \leq 1, j \in J. \end{array}$$

We now establish the main result of this section which shows that in some special cases, the optimal solution of the $l_1$ norm inverse problem of problem (LP) can be obtained from the dual optimal solution.

**Theorem 2.2.1.** Suppose $x^0$ is a given 0-1 feasible solution of problem (LP) which has an optimal solution $x^*$ satisfying $0 \leq x^* \leq 1$. Let $\pi^*$ be the optimal solution of its dual problem (DLP). Define vectors $\theta^* = 0$ and $\alpha_j^* = max\{0, c_j - \pi^* p_j\}$ for all $j \in J$. Then $\{\pi^*, \theta^*, \alpha^*\}$ is an optimal solution of the inverse problem (ILP1).

*Proof.* The dual of problem (LP) is

(DLP)

$$\begin{array}{ll} \max & \pi b \\ \text{s.t.} & \pi p_j \leq c_j, j = 1, 2, ..., n. \end{array}$$

We first show that $\{\pi^*, \theta^*, \alpha^*\}$ is a feasible solution to problem (ILP1). Obviously, $\theta^*, \alpha^* \geq 0$. As $\pi^*$ is a feasible solution of problem (DLP), for any $j \in \underline{\mathrm{J}} \cup J$,

$$-\pi^* p_j + \theta_j^* = -\pi^* p_j \geq -c_j.$$

Also, by the definition of $\alpha_j^*$, for $j \in J$.

$$\pi^* p_j + \alpha_j^* = \pi^* p_j + max\{0, c_j - \pi^* p_j\} \geq c_j.$$

So, $\{\pi^*, \theta^*, \alpha^*\}$ is feasible.

Note that the objective value of problem (ILP1) is $\sum_{j \in J} \alpha_j^*$.

We now prove that $\{\pi^*, \theta^*, \alpha^*\}$ is an optimal solution. In order to do so it suffices if we can show that the dual problem (DILP1) has a feasible solution with the same objective value $\sum_{j \in J} \alpha_j^*$.

Since $x^*$ and $\pi^*$ are respectively the optimal solutions of (LP) and (DLP), by the complementary slackness condition,

$$\sum_{j \in \underline{\mathrm{J}} \cup J} (c_j - \pi^* p_j) x_j^* = 0. \tag{2.6}$$

As $x^0$ is a $0-1$ vector, for each $j \in J$, $x_j^0 = 1$. By the definition of $\alpha_j^*$ and the feasibility of $\pi^*$ to problem (DLP), for each $j \in J$, $\alpha_j^* = c_j - \pi^* p_j$. So,

$$\begin{aligned}\sum_{j\in J}\alpha_j^* &= \sum_{j\in J}\alpha_j^* x_j^0\\ &= \sum_{j\in J}(c_j - \pi^* p_j)x_j^0\\ &= \sum_{j\in \underline{\text{J}}\cup J}(c_j - \pi^* p_j)x_j^0\end{aligned}$$

Combining Eqs. (2.6) and above equation., we obtain

$$\begin{aligned}\sum_{j\in J}\alpha_j^* &= \sum_{j\imath\underline{\text{J}}\cup J}(c_j - \pi^* p_j)(x_j^0 - x_j^*)\\ &= -c^T(x^* - x^0) + \pi^* A(x^* - x^0)\\ &= -c^T(x^* - x^0).\end{aligned}$$

since

$$x_j^* - x_j^0 = \begin{cases} x_j^* \in [0,1], \quad if \quad j \in \underline{\text{J}},\\ x_j^* - 1 \in [-1,0], \quad if \quad j \in J,\end{cases}$$

$x^* - x^0$ is a feasible solution of problem (DILP1) with the objective value $-c^T(x^* - x^0)$. Therefore, $(\pi^*, \theta^*, \alpha^*)$ and $x^* - x^0$ are, respectively, the optimal solutions of problems (ILP1) and (DILP1). □

## 2.3 The solution of (IBLP) under $l_1$ norm

Under the $I_1$ norm, the inverse problem (ILBP) becomes

(IBLP1)

$$\begin{aligned} \text{Min} \quad & \textstyle\sum_{j\in \underline{J}\cup J}\theta_j + \sum_{j\in J\cup \bar{J}}\alpha_j \\ \text{s.t.} \quad & -\pi p_j + \theta_j \geq -c_j, j\in \underline{J}\cup J, \\ & \pi p_j + \alpha_j \geq c_j, j\in J\cup \bar{J}, \\ & \theta_j \geq 0, j\in \underline{J}\cup J, \\ & \alpha_j \geq 0, j\in J\cup \bar{J} \end{aligned}$$

Let $A_1$ and $A_2$ be the submatrices consisting of the columns $p_j$ of A, corresponding to $j \in \underline{J}\cup J$ and $j \in J\cup\bar{J}$, respectively. Then the dual of the above problem is

$$\begin{aligned} \text{Max} \quad & -\textstyle\sum_{j\in \underline{J}\cup J} c_j x_j + \sum_{j\in J\cup \bar{J}} c_j y_j \\ \text{s.t.} \quad & A_1x - A_2y = 0, \\ & 0 \leq x_j \leq 1, \quad j\in \underline{J}\cup J, \\ & 0 \leq y_j \leq 1, \quad j\in J\cup \bar{J}. \end{aligned}$$

In what follows we consider a special case of the problem (BLP): $u = 1$, i.e. each variable $x_j$ has a unit upper-bound: $u_j = 1$. Problem (BLP) becomes

(UBLP)

$$\begin{aligned} \text{Min} \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & 0 \leq x \leq 1. \end{aligned}$$

Its dual is

(DUBLP)

$$\begin{aligned} \text{Max} \quad & \pi b - \omega l \\ \text{s.t.} \quad & \pi A - \omega \leq C^T, \\ & \omega \geq 0, \end{aligned}$$

in which $\pi \in R^m$ and $\omega \in R^n$ are two row vectors. Let $x^0$ be a 0-1 feasible solution of problem (UBLP), then the inverse problem with respect to $x^0$ is (IUBLP1)

$$\begin{aligned} \text{Min} \quad & \textstyle\sum_{j \in \underline{J}} \theta_j + \sum_{j \in \bar{J}} \alpha_j \\ \text{s.t.} \quad & -\pi p_j + \theta_j \geq -c_j, \quad j \in \underline{J}, \\ & \pi p_j + \alpha_j \geq c_j, \quad j \in \bar{J}, \\ & \theta_j \geq 0, \quad j \in \underline{J}, \\ & \alpha_j \geq 0, j \in \bar{J}, \end{aligned}$$

or equivalently,

(IUBLP1')

$$\begin{aligned} \text{Min} \quad & \textstyle\sum_{j \in \underline{J}} \theta_j + \sum_{j \in \bar{J}} \alpha_j \\ \text{s.t.} \quad & -\pi p_j + \theta_j \geq -c_j, j \in \underline{J} \\ & -\pi p_j - \alpha_j + \omega_j = -c_j, \quad j \in \bar{J} \\ & \theta_j \geq 0, j \in \underline{J}, \\ & \alpha_j, \quad \omega_j \geq 0, j \in \bar{J}. \end{aligned}$$

Obviously the dual of (IUBLP1) is

(DIUBLP1)

$$\begin{aligned} \text{Max} \quad & -\textstyle\sum_{j \in \underline{J}} c_j x_j + \sum_{j \in \bar{J}} c_j y_j \\ \text{s.t.} \quad & A_{\underline{J}} x - A_{\bar{J}} y = 0, \\ & 0 \leq x_j \leq 1, j \in \underline{J}, \\ & 0 \leq y_j \leq 1, j \in \bar{J}. \end{aligned}$$

Or, if we define $x_j = -y_j$ for $j \in \bar{J}$, then the dual of the inverse problem can be expressed as

(DIUBLP1')

$$\begin{aligned} \text{Max} \quad & -c^T x \\ \text{s.t.} \quad & Ax = 0 \\ & 0 \le x_j \le 1, j \in \underline{\mathrm{J}}, \\ & -1 \le x_j \le 0, j \in \bar{J}. \end{aligned}$$

We now give a very simple method to obtain an optimal solution of the inverse problem (IUBLP1).

# Chapter 3

# The Inverse Optimal Value Problem

## 3.1 Definitions and assumptions

Consider the optimal value function of a linear program in terms of its cost vector

$$Q(c) := min_x\{c^T x | Ax = b, x \geq 0\}, \quad (1)$$

where $x \in R^n$. Given a set $C \subseteq R^n$ of the objective cost vectors and a real number $z^*$, this project is concerned with the inverse optimization problem of finding a cost vector from the set $C$ such that the optimal objective value of the linear program (1) is "close" to $z^*$. The problem can be formulated as

$$min_c\{f(c) | c \in C\}, \quad (2)$$

where $f(c) := |Q(c) - z^*|$ if $Q(c) \in R$ and $f(c) := +\infty$ if $Q(c) \in \{-\infty, +\infty\}$. we refer to(2) as the inverse optimal value problem. Note that an instance of (2) is given by specifying the linear programming value function $Q$, the set of feasible cost vectors $C$, and the desired optimal objective value $z^*$. We shall denote such an instance by $P(Q, C, z^*)$.

Throughout the rest of this project, we make the following assumptions:

(A1) The feasible region of the linear program $\{x|Ax = b, x \geq 0\}$ is non-empty.

(A2) The set of cost vectors $C$ is non-empty, compact, and convex.

(A3) $C \cap C_\infty \neq \emptyset$.

By assumption (A1), we have that $Q : R^n \to [-\infty, +\infty)$.Using strong duality, we can write

$$Q(c) = max_\pi\{\pi^T b|\pi^T A \leq c\}, \quad (3)$$

and also $C_{z^*} = \{c|\exists \ \pi \ \ s.t.\pi^T A \leq c, \pi^T b \geq z^*\}$ and $C_\infty = \{c| \ \exists \ \ \pi s.t.\pi^T A \leq c\}$.

The following properties are easily verified.

**Proposition 3.1.1.** (i) $Q(.)$ is upper-semi-continuous over $R^n$.

(ii) $Q(.)$ is piece-wise linear, concave and continuous over $C_\infty$.

(iii) The sets $C_z^*$ and $C_\infty$ are closed and convex

Furthermore, the non-negativity restriction in the linear program (1)

**Proposition 3.1.2.** $Q(.)$ is non-decreasing over $R^n$.

Finally, since $f(.)$ is continuous over the non-empty compact set $C \cap C_\infty$,

**Proposition 3.1.3.** The inverse optimal value problem (2) has a finite optimal solution.

## 3.2 Complexity

Given an integer matrix $B \in Z^{m\times n}$, and an integer vector $d \in Z^m$, the binary integer feasibility problem can be stated as follows
An instance of the binary integer feasibility problem is specified by the matrix vector pair $(B, d)$. We shall denote such an instance by $\mathcal{B}(B, d)$.

**Lemma 3.2.1.** Given an instance $\mathcal{B}(B, d)$, we can construct an instance $P(\hat{Q}, \hat{C}, \hat{z^*})$ of the inverse optimal value problem, such that $\mathcal{B}(B, d)$ has an answer "yes" if and only if the optimal objective value of $P(\hat{Q}, \hat{C}, \hat{z^*})$ is zero.

*Proof.* Given an instance $\mathcal{B}(B, d)$ with $B \in Z^{m\times n}$, and $d \in Z^m$, let us define the compact polyhedral set

$$\begin{aligned}\hat{C} := \Big\{ &(c_1, c_2, c_3)^T \in R^{3n} | c_1 \in R^n, c_2 \in R^n, c_3 \in R^n, \\ &Bc_1 \leq d, c_1 = c_2, c_3 = e, \\ &0 \leq c_1 \leq e, 0 \leq c_2 \leq e \Big\}\end{aligned}$$

and the linear programming value function $\hat{Q} : R^{3n} \to R$ as:

$$\begin{aligned}\hat{Q}(c) := min \;\; & c_1^T u - c_2^T v + c_3^T v \\ s.t. \;\; & u + v \geq e, u \in R^n_+, v \in R^n_+,\end{aligned}$$

where $e \in R^n$ is a vector of ones. Finally, letting $\hat{z}^* = 0$, we have an instance $P(\hat{Q}$, $\hat{C}$ ,$\hat{z}^*$ of the inverse optimal value problem.

Suppose $\mathcal{B}(B, d)$ has an answer "yes," i.e., there exists $\hat{x} \in \{0, 1\}^n$ such that

$B\hat{x} \leq d$. Consider a cost vector $\hat{c} = (c_1, c_2, c_3)^T$ such that $c_1 = c_2 = \hat{x}$ and $c_3 = e$. Clearly $\hat{c} \in \hat{C}$. Now, note that

$$\hat{Q}(\hat{c}) := \sum_{j=1}^{n} \quad min \;\; \hat{x}_j u_j + (1 - \hat{x}_j)vj$$
$$s.t. u_j + v_j \geq 1, u_j, v_j \geq 0$$

Since $\hat{x}_j \in \{0, 1\}$, we have $\hat{x}_j = 0$ implies $v_j = 0$, and $\hat{x}_j = 1$ implies $u_j = 0$. Thus $\hat{Q}(\hat{c}) = 0 = \hat{z}^*$ and the optimal objective function in $P(\hat{Q}, \hat{C}, \hat{z}^*)$is zero. Now suppose the optimal objective value in $P(\hat{Q}, \hat{C}, \hat{z^*})$ is zero, i.e., there exists $\bar{c} \in \hat{C}$ such that $\hat{Q}(\hat{c}) = 0$. Let $\bar{c} := (\hat{c}, \hat{c}, e)^T$, where $\hat{c} \in R^n$. Note that

$$\hat{Q}(\bar{c}) = \textstyle\sum_{j=1}^{n} \hat{Q}_j(\hat{c}),$$

where

$$\hat{Q}_j(\hat{c}) = min \;\; \hat{c}_j uj + (1 - \hat{c}_j)vj$$
$$s.t. u_j + v_j \geq 1, u_j, v_j \geq 0.$$

Since $0 \leq \hat{c}_j \leq 1$, the optimal value of the above linear program will satisfy $\hat{Q}_j(\hat{c}) = min\{\hat{c}_j, 1 - \hat{c}_j\}$ for all $j$. Furthermore, $\hat{Q}(\bar{c}) = 0$ implies $\hat{Q}_j(\hat{c}) = 0$ for all $j$. It then follows that $\hat{c}_j \in \{0, 1\}$ for all $j$. Then, from the fact that $(\hat{c}, \hat{c}, e)^T \in \hat{C}$, we have that the binary vector $x = \hat{c}$ provides an affirmative answer for $\mathcal{B}(B, d)$. □

**Theorem 3.2.1.** The inverse optimal value problem is NP-hard.

*Proof.* Lemma 3.2.1 shows that we can provide an answer to any binary integer feasibility question by constructing and solving an equivalent instance of the inverse optimal value problem. The claim follows from the fact that the binary integer feasibility problem is NP-complete, and that the construction in Lemma 3.2.1 is clearly polynomial. □

## 3.3 Structural Results

we describe some structural conditions to reduce the inverse optimal value problem to well-known optimization problems. Our analysis centers on whether the set $\bar{C}$ is empty or non-empty.

**Proposition 3.3.1.** Suppose $\bar{C} = \phi$. Let $c^*$ be an optimal solution of

$$max\{Q(c) \quad |c \in C\}, \qquad (4)$$

then $c^*$ is an optimal solution of the inverse optimal value problem (2).

*Proof.* Since $Q(c)$ is upper-semi-continuous over the non-empty, convex, and compact feasible region $C$, problem (4) has a well-defined optimal solution. Since $\bar{C} = \phi$,it follows that $Q(c) < z^*$ for all $c \in C$. Problem (2) then reduces to (4). Using the dual representation (3) of $Q(c)$, we can state problem (4) in the above proposition as:

$$\begin{aligned} \max_{c,\pi} \quad & b^T\pi \\ \text{s.t.} \quad & c \in C, \\ & \pi^T A - c \leq 0. \qquad (5) \end{aligned}$$

The above problem involves maximizing a linear function over a convex set for which a variety of efficient algorithms exist. If $C$ is polyhedral, problem (5) is simply a linear program. □

**Proposition 3.3.2.** Suppose $\bar{C} \neq \phi$. Let $c^*$ be an optimal solution to

$$min\{Q(c) \quad | c \in \bar{C}\}, \qquad (6)$$

then $c^*$ is an optimal solution of the inverse optimal value problem (2).

**Lemma 3.3.1.** Given any point $c^* \in \bar{C}$, there exists $c' \in [\bar{c}^L, c^*]$ such that $c' \in \partial_L \bar{C}$.

**Proposition 3.3.3.** If $\bar{C} \neq \phi$ , then there exists an optimal solution $c^*$ of the inverse optimal value problem (2) such that $c^* \in \partial_L \bar{C} \cap \Omega(\bar{C})$.

*Proof.* We first argue that the set $\partial_L \bar{C} \cap \Omega(\bar{C})$ is non-empty.
Let $S = argmin\{e^T c \quad | c \in \bar{C}\}$.
We claim that $S \subseteq \partial_L \bar{C}$.
Suppose it is not true .
Consider $c' \in S \setminus \partial_L \bar{C}$. Since $c' \in \bar{C}$,
by Lemma 3.3.1, there exists $c'' \in \partial_L \bar{C}$ such that $c'' < c'$, thus $e^T c'' < e^T c'$, and $c'$ cannot be in $S$. Since $S \cap \Omega(\bar{C}) \neq \phi$, we have $\partial_L \bar{C} \cap \Omega(\bar{C}) \neq \phi$.
Now note that by Proposition 3.3.2, problem (2) is equivalent to problem (6).
Consider an optimal solution $c^*$ to problem (6) such that $c^* \notin \partial_L \bar{C} \cap \Omega(\bar{C})$. By Lemma 3.3.1, there exists $c' < c^*$ such that $c' \in \partial_L \bar{C}$.
By the non-decreasing property of $Q(.)$, we have $Q(c') \leq Q(c^*)$, therefore $c'$ is also an optimal solution.
By convexity of $\bar{C}$, we can write $c' = \sum_{i \in I} \lambda_i c_i + (1 - \sum_{i \in I} \lambda_i) c_0$ where $I$ is an appropriate index set, $c_i \in \Omega(\bar{C})$ and $\lambda_i \geq 0$ for $i \in I$,$\sum_{i \in I} \lambda_i \leq 1$, and $c_0 \in \partial_L \bar{C} \cap \Omega(\bar{C})$.

Using concavity of $Q$ we have that $Q(c') \geq \sum_{i \in I} \lambda_i Q(c_i) + (1 - \sum_{i \in I} \lambda_i) Q(c_0)$. Since $Q(c') \leq Q(c_i)$ for all $i \in I$, we have that $Q(c_0) \leq Q(c')$, thus $c_0 \in \partial_L \bar{C} \cap \Omega(\bar{C})$ is also an optimal solution for the problem. □

## 3.4 Conditions for polynomial solvability

we state more general conditions guaranteeing easy solvability of the inverse optimal value problem. In addition to assumptions (A1)-(A3), we shall also require the following assumption
$(A4) C \subseteq C_\infty$.
Assumption (A4) guarantees that the underlying linear program is bounded for all cost vectors in $C$. The assumption is trivially satisfied when the feasible region $\{x \quad | Ax = b, x \geq 0\}$ of the underlying LP is bounded.

**Proposition 3.4.1.** Suppose $\bar{C} \neq \phi$ and $\bar{c}^L \in C$. Let $c^*$ be an optimal solution to the following problem

$$min\{e^T c | c \in \bar{C}\}. \qquad (7)$$

Then $c^*$ is an optimal solution to problem (6) and, hence, is an optimal solution to the inverse optimal value problem (2).

*Proof.* Consider first the case when $\bar{c}^L \in \bar{C}$. Then from Proposition 3.3.3, it follows that $\bar{c}^L$ is an optimal solution to problem (6) (since $\partial^L \bar{C} \cap \Omega(\bar{C}) = \{\bar{c}^L\}$) and,hence, is an optimal solution to the inverse optimal value problem (2). The claim then follows from noting that in this case $c^* = \bar{c}^L$ is the unique optimal solution of (7).

Now consider the case that $\bar{c}^L \notin \bar{C}$. Suppose that the claim is not true. Then $z^* < Q(c^*)$. Since $\bar{c}^L \notin \bar{C}$, we have $-\infty < Q(\bar{c}^L) < z^* < Q(c^*)$ where the first inequality is a consequence of Assumption (A4). Since $\bar{c}^L \in C$, we have that $[\bar{c}^L, c^*] \in C \cap C_\infty$. By Proposition 3.1.1(ii) $Q(.)$ is continuous over $[\bar{c}^L, c^*]$, so there exists $c' \in (\bar{c}^L, c^*)$ such that $Q(c') = z^*$. Then $c' \in \bar{C}$ and, since clearly $c' < c^*$, we have $e^T c' < e^T c^*$. Therefore $c^*$ cannot be an optimal solution to (7). Problem (7) above is equivalent to

$$\begin{aligned} \min_{c,\pi} \quad & e^T c, \\ \text{s.t.} \quad & c \in C, \\ & \pi^T A - c \leq 0, \pi^T b \geq z^*, \end{aligned} \tag{8}$$

and is a convex program with a linear objective, and can be solved quite efficiently.In particular, when C is polyhedral, problem (8) is simply a linear program. □

**Theorem 3.4.1.** If $\bar{c}^L \in C$, and the convex programs (4) and (7) can be solved in polynomial time, then the inverse optimal value problem (2) can be solved in polynomial time.

*Proof.* If (7) can be solved in polynomial time, then we can verify whether the convex set $\bar{C} = \phi$ in polynomial time. If $\bar{C} \neq \phi$, then by Proposition 3.4.1, an optimal solution of (8) is an optimal solution of (2). If $\bar{C} = \phi$, Proposition 3.3.1 implies that an optimal solution to (2) can be found in polynomial time by solving the convex program (4). □

# Chapter 4

# Applications to inverse network optimisation

We know that the shortest path problem from node $s$ to $t$ in a network $N(V, A, C)$ can be formulated as a LP problem:

$$\begin{array}{ll} \text{Min} & \sum_{(i,j)\in A} c_{ij}x_{ij} \\ \text{s.t.} & -\sum_{(i,j)\in A} x_{i,j} + \sum_{(k,i)\in A} x_{ki} = \begin{cases} -1, & i = s, \\ 0, & i \in V \setminus \{s,t\}, \\ 1, & i = t, \end{cases} \quad x_{i,j} \geq 0, (i,j) \in A \end{array} \tag{4.1}$$

Note that as the coefficient matrix is the node-edge incidence matrix of the network, which is totally unimodular, each basic feasible solution must be a 0-1 solution. So, a $0-1$ optimal solution $x^*$ exists in which the components with $x^*_{ij} = 1$ correspond to a shortest path. Now suppose a path $P$ from $s$ to $t$ is given. By defining

$$x_{ij}^0 = \begin{cases} 1, (i,j) \in P, \\ 0, otherwise. \end{cases}$$

We have a $0 - 1$ feasible solution $x^0$ to the above LP problem and thus the conditions of Theorem 2.2.1 are satisfied. Now the optimal solution of the inverse shortest path problem in the sense of this project can be obtained very quickly by using Theorem 2.2.1. In fact the dual of problem (4.1) is

$$\begin{aligned} &Max\ \pi_t - \pi_s \\ &s.t\ \pi_j - \pi_i \leq c_{ij},\ (i,j) \in A, \end{aligned}$$

and it is well known that $\pi_i$ represents the shortest length from $s$ to $i$ (possibly plus a constant). Therefore, the algorithm for such type of inverse shortest path problems consists of the following three steps:

Step 1. Find the shortest distance $\pi_i^*$ from $s$ to each node $i \in V$.

Step 2. For each $(i,j) \in P$, define

$\alpha_{ij}^* = c_{ij} + \pi_i^* - \pi_j^*$.

Step 3. Let

$$\bar{c_{ij}} = \begin{cases} c_{ij} - \alpha_{ij}^*, (i,j) \in P, \\ c_{ij}, otherwise, \end{cases}$$

then $\bar{c}$ is the least-change (under $l_1$ measure) cost vector to make $P$ become a shortest path from $s$ to $t$.

Note that when there is a negative cycle in the network, it may not have a shortest path from $s$ to $t$. But the inverse problem is still solvable, because we can insert the constraints

$x_{ij} \leq 1$, $(i,j) \in A$,

to problem (4.1), and then to solve the inverse BLP problem.

If the $l_\infty$ measure is concerned, for inverse shortest path problem we can first solve the LP problem:

$$\begin{aligned} \text{Min} \quad & \mathbf{v} \\ \text{s.t.} \quad & \pi_i - \pi_j + \mathbf{v} \geq -c_{ij}, (i,j) \in A, \\ & \pi_j - \pi_i + \mathbf{v} \geq c_{ij}, (i,j) \in P, \end{aligned}$$

obtaining an optimal solution $(\pi^*, \mathbf{v}^*)$, and then the least-change cost vector under $l_\infty$ norm is

$$\bar{c}_{ij} = \begin{cases} c_{ij} + \mathbf{v}^*, (i,j) \in A \quad \pi^*{}_j - \pi^*{}_i > c_{ij}, \\ c_{ij} - \mathbf{v}^*, (i,j) \in P \quad \pi_j{}^* - \pi_i{}^* < c_{i,j}, \\ c_{ij}, otherwise. \end{cases}$$

The inverse minimum spanning tree problem under $l_\infty$ norm is especially simple to solve.

Let $N = (V, E, c)$ be a undirected network and $T$ be a given spanning tree in $N$. For each $e \in E \setminus T$, $T \cup \{e\}$ contains a unique cycle. It is easy to know that in order to let $T$ become a minimum spanning tree, we only need to reduce the weights on $T$ and increase the weights not on $T$. In particular, the inverse problem under $l_\infty$ measure can be formulated to:

$(IST\infty)$

$$\begin{aligned} \text{Min} \quad & max\{\theta_e, \alpha_j\} \\ \text{s.t.} \quad & c_e + \theta_e \geq c_f - \alpha_f, e \notin T, f \in C(T,e), \\ & \theta_e \geq 0, e \notin T, \\ & \alpha_f \geq 0, f \in T, \end{aligned}$$

where $C(T,e)$ consists of the subset of edges in $T$ which together with $e$ form the unique cycle in $T \cup \{e\}$.

Using the method , we can change problem $(IST\infty)$ to

$$\begin{aligned} \text{Min} \quad & \mathbf{v} \\ \text{s.t.} \quad & 2\mathbf{v} \geq c_f - c_e, e \notin T, f \in C(T,e), \\ & \mathbf{v} \geq 0, \end{aligned}$$

without affecting the optimal value. Obviously, the optimal solution (value) of the above problem is

$$\mathbf{v}^* = (\tfrac{1}{2})max\{0, max_{e \notin T} \;\; max_{f \in C(T,e)}\{c_f - c_e\}\},$$

and the least-change cost vector $\bar{c}$ under the $l_\infty$ norm which lets $T$ become the minimum spanning tree in $N$ is

$$\bar{c_g} = \begin{cases} c_g - \mathbf{v}^*, g \in T, \\ c_g + \mathbf{v}^*, g \notin T, \end{cases}$$

for each edge $g$.

# Conclusion

Inverse linear programming is a versatile tool with real- world applications across various domains. Its ability to solve complex optimization problems makes it invaluable for businesses and organizations seeking efficient , cost- effective solutions. Inverse linear programming offers a valuable approach for addressing various optimization problems where the objective function is known, but the constraints need to be inferred. By reversing the traditional optimization process, inverse linear programming aims to determine the set of constraints that lead to a given optimal solution. This methodology finds applications in diverse fields such as economics, engineering, operations research, and finance, where decision-makers seek to understand the underlying constraints governing a given optimal outcome. Through techniques such as sensitivity analysis, feasibility analysis, and convex optimization methods, inverse linear programming provides insights into the structure and boundaries of feasible regions, facilitating better decision-making and resource allocation. Furthermore, advancements in computational algorithms and mathematical modeling have enhanced the efficiency and scalability of inverse linear programming techniques, enabling their application to larger and more complex optimization problems. Overall, inverse linear programming serves as a valuable tool for reverse-engineering optimization solutions, offering practical solutions to real-world decision-making challenges.

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# Coding Theory

**Project report submitted to Christ College (Autonomous) in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics**

by

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# CERTIFICATE

This is to certify that the project entitled **"Coding Theory"** submitted to Postgraduate and Research Department of Mathematics in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics, is a bonafide record of project work done by **Ms.NEERAJA T D(CCAWMMS015)** during the period of her study in the Postgraduate and Research Department of Mathematics, Christ College (Autonomous), Irinjalakuda, under my supervision and guidance during the year 2022-2024

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# DECLARATION

I hereby declare that the project work entitled "**Coding theory**" submitted to Christ College(Autonomous), Irinjalakuda in partial fulfilment of the requirement for the award of Master Degree of Science in Mathematics is a record of project work done by me during the period of my study in the Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda.

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First, there are no words to adequately acknowledge the wonderful grace that my Redeemer has given me. There are many individuals who have come together to make this project a reality. I greatly appreciate the inspiration; support and guidance of all those people who have been instrumental for making this project a success.

I express my deepest thanks to my guide Ms Tintumol Sunny, Assistant Professor, Department of Mathematics, Christ College(Autonomous), Irinjalakuda, who guided me faithfully through this entire project. I have learned so much from her, both in the subject and otherwise. Without her advice, support and guidance, it find difficult to complete this work.

I take this opportunity to express my thanks to our beloved principal Fr. Dr. Jolly Andrews CMI, who gave me the golden opportunity to do this wonderful project on the topic "**Coding Theory**"

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I want to especially thank all the faculty of the library for providing various

facilities for this project.

Finally, I would like to thank my family who motivated me and boosted my morale whan I was stressed.

**Neeraja T D**

# Contents

# Introduction

Coding theory is the study of the properties of codes and their respectives fitness for specific applications. Codes are used for data compression, cryptography, error detection and correction, data transmission and data storage. Codes are studied by various scientific disciplines-such as information theory, electrical engineering, mathematics, linguistics, and computer science-for the purpose of designing efficient and reliable data transmission methods. This typically involves the removal of redundancy and the correction or detection of errors in the transmitted data.

Coding theory is an application of information theory critical for reliable communication and fault-tolerant information storage and processing; indeed, the shannon channel coding theorem tells us that we can transmit information on a noisy channel with an arbitrarily low probability of error. A code is designed based on a well-defined set of specifications and protects the information only for the type and number of errors prescribed in its design. Shannon developed information entropy as a measure for the uncertainty in a message while essentially inventing the field of information theory. The binary Golay code was developed in 1949. It is an error-correcting code capable of correcting up to three errors in each 24-bit word, and detecting a fourth.

Error detection and correction codes have evolved over several decades. In 1948, Claude Shannon's landmark paper, titled "A Mathematical Theory of Communication," perhaps started the formal discipline of coding theory. Working at Bell Labs, Shannon showed that it was possible to encode messages for transmission in such a way that the number of extra bits was minimal. Few years later, Richard Hamming, also in Bell Labs, produced a 3-bit code for four data bits. Hamming invented this code after several failed attempts to punch out a message on a paper using the parity code. Apparently, Hamming expressed his frustration in the following words, "If it can detect the error, why can't it correct it!" Since the early 1950s, coding theory has evolved to cover a variety of fault models and situations. Today, error detection and correction codes are widely used across various forms of computing systems, including the ones sent for space exploration.

## Outline of the Project

Apart from the introductory chapter, we have described our work in four chapters.

In **Chapter 1**, we discussed basic concepts of coding theory. It is the study of methods for efficient and accurate transfer of information from one place to another. We also discussing on encoding and decoding, error-detecting and correcting codes.

In **Chapter 2**, we are discussing about Linear codes and their distance. We are discussing on its application on matrices.

In **Chapter 3**, We turn our attention to the problem of determining how many words a linear code of given length have. We are discussing about perfect codes, hamming codes, extended codes and extended golay codes.

In **Chapter 4**, A Reed-Solomon code is an error-correcting code first described by Reed and Solomon. Here we are discussing on encoding and implementation of Reed-Solomon coding.

# Riemann Manifolds

Project report submitted to Christ College (Autonomous) in partial fulfillment of the requirement for the award of the M.Sc Degree programme in Mathematics

by

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**2024**

# CERTIFICATE

This is to certify that the project entitled **"Riemann Manifolds"** submitted to Postgraduate and Research Department of Mathematics in partial fulfilment of the requirement for the award of the M.Sc Degree programme in Mathematics, is a bonafide record of project work done by **Mr. ROSH KRISHNA V (CCAWMMS016)** during the period of his study in the Postgraduate and Research Department of Mathematics, Christ College (Autonomous), Irinjalakuda, under my supervision and guidance during the year 2023-2024.

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I hereby declare that the project work entitled "**Riemann Manifolds**" submitted to Christ College(Autonomous), Irinjalakuda in partial fulfilment of the requirement for the award of Master Degree of Science in Mathematics is a record of project work done by me during the period of my study in the Postgraduate and Research Department of Mathematics, Christ College(Autonomous), Irinjalakuda.

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# ACKNOWLEDGEMENT

First, there are no words to adequately acknowledge the wonderful grace that my Redeemer has given me. There are many individuals who have come together to make this project a reality. I greatly appreciate the inspiration; support and guidance of all those people who have been instrumental for making this project a success.

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facilities for this project.

Words cannot express the love and support I have received from my parents, whose encouragement has buoyed me up from the beginning till the end of this work.

Rosh Krishna V

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