22U505 (Pages: 2) Name: ……..……….……..

Reg. No: ………………….

FIFTH SEMESTER B.Sc. DEGREE EXAMINATION, NOVEMBER 2024

(CBCSS-UG)

(Regular/Supplementary/Improvement)

CC20U MTS5 B05 – ABSTRACT ALGEBRA

(Mathematics – Core Course)

(2020 Admission onwards)

Time: 2 ½ Hours Maximum: 80 Marks

Credit: 4

Part - A

Answer ***all*** questions. Each question carries 2 marks.

1. Find $\varphi(10)$ and $\varphi(42)$.
2. Find the product of cycles (1, 4, 2, 5) and (2, 6, 3).
3. Define an abelian group.
4. Give examples of (a) an infinite cyclic group and (b) a finite cyclic group.
5. Describe Klein four-group.
6. Show that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.
7. Find the order of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ in $GL_2(\mathbb{R})$.
8. Define transposition. Express (1,2,3) (4,5) as a product of transpositions.
9. Show that the kernel of a homomorphism is one–to–one if and only if kernel = {e}.
10. Prove that the general linear group $GL_n(F)$ is a group under matrix multiplication.
11. Show that subgroups of index 2 are normal.
12. Determine whether or not $*$ gives a group structure on the set $\mathbb{Z}$ defined by $a * b = ab$.
13. Define a simple graph and give an example.
14. Define the unit in a ring R. Find the units in the ring $\mathbb{Z}_6$.
15. Give an example of an integral domain that is not a field. Justify your answer.

(Ceiling: 25 Marks)

Part - B

Answer ***all*** questions. Each question carries 5 marks.

16. Consider the set of all differentiable functions from R to R. Define a relation '$\sim$' on this set by $f \sim g$ if and only if the derivatives $f'(x)$ and $g'(x)$ are equal for all $x \in R$. Show that '$\sim$' is an equivalence relation on the set of differentiable functions.
17. Show that the set $\mathbb{Z}_n$ of integers modulo n is an abelian group under the addition of congruence classes.

18. Draw the subgroup diagram of $\mathbb{Z}_{15}$.

19. Describe the dihedral group D_3.

20. If $\varphi: G_1 \to G_2$ is a group homomorphism, then prove the following:

    a) $\varphi(e) = e$ b) $(\varphi(a))^{-1} = \varphi(a^{-1})$ for all $a \in G_1$.

21. Prove that any permutation in S_n, where n > 2, can be written as a product of transpositions.

22. Prove that for group G, then Aut(G) is a group under the composition of functions and Inn(G) is a normal subgroup of Aut(G).

23. Show that if D is an integral domain, then D[x] is also an integral domain.

(Ceiling: 35 Marks)

PART- C (Essay Type)

Answer any ***two*** questions. Each question carries 10 marks.

24. a) Prove that if (a, n) = 1, then $a^{\varphi(n)} \equiv 1(\text{mod } n)$.

    b) Prove that if p is a prime number then for any integer 'a' we have $a^p \equiv a(\text{mod } p)$.

25. a) State and prove Lagrange's theorem.

    b) Prove that any group of prime order is cyclic.

26. Prove that every subgroup of a cyclic group is cyclic.

27. a) State and prove the second isomorphism theorem.

    b) Show that $\mathbb{Z}_n/m\mathbb{Z}_n \cong \mathbb{Z}_m$ if $m|n$.

(2 × 10 = 10 Marks)

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