24P105 (Pages: 2) Name: ....

Reg.No: ....

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2024

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH1 C05 - NUMBER THEORY

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Find all integers n such that $\phi(n) = \frac{n}{2}$.
2. Prove that the Mobius function is multiplicative but not completely multiplicative.
3. If f and g are arithmetical functions and let $h = f * g, H(x) = \sum\limits_{n \leq x} h(n), F(x) = \sum\limits_{n \leq x} f(n)$ and $G(x) = \sum\limits_{n \leq x} g(n)$.vThen show that $H(x) = (f \circ G)(x) = (g \circ F)(x)$
4. Prove that $\lim\limits_{n \to \infty} \frac{P_n}{n \log n} = 1$ implies $\lim\limits_{x \to \infty} \frac{\pi(x) \log \pi(x)}{x} = 1$, where P_n denotes the n^{th} prime.
5. Prove that $\forall\ x \geq 1,\ \sum\limits_{n \leq x} \tau(\frac{x}{n}) = x \log x + O(x)$.
6. Define Shift cryptosystem. Find the plain text of the cipher text 'HPHTWWXPPE' in the shift cryptosystem with $b = 11$ and $N = 26$.
7. Find the inverse of the matrix $\begin{bmatrix} 197 & 62 \\ 603 & 271 \end{bmatrix} \pmod{841}$.
8. How will you authenticate a message in public key cryptosystem?

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT - I

9. If f is an arithmetical function with $f(1) \neq 0$, then prove that there exists a unique arithmetical function f^{-1} such that $f * f^{-1} = I$.
   Also show that $f^{-1}(1) = \frac{1}{f(1)}$ and $f^{-1}(n) = \frac{-1}{f(1)} \sum\limits_{\substack{d/n \\ d<n}} f(\frac{n}{d}) f(d),\ \forall n > 1$.
10. Derive the divisor sum of Mangoldt function and then deduce that $\forall n \geq 1,\ \Lambda(n) = \sum\limits_{d/n} \mu(d) \log(\frac{n}{d})$.
11. State and prove Selberg identity.

UNIT - II

12. Show that for $x > 0,\ 0 \leq \frac{\psi(x)}{x} - \frac{\tau(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x}\log 2}$

13. Prove that for $x \geq 2, \tau(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt$ and $\pi(x) = \frac{\tau(x)}{\log x} + \int_2^x \frac{\tau(t)}{t\log^2 t}dt.$

14. Show that there is a constant A such that $\sum_{p\leq x} \frac{1}{p} = \log\log x + A + O(\frac{1}{\log x}),\ \forall\ x \geq 2.$

UNIT - III

15. Let p be an odd prime. Then prove that $\forall\ n,\ (n|p) \equiv n^{\frac{p-1}{2}} (\mod p).$

16. For every odd prime p, prove that $(2|p) = (-1)^{\frac{p^2-1}{8}}$.

17. Solve the system: $x + 3y \equiv 1(\text{mod } 26)$
$7x + 9y \equiv 1(\text{mod } 26)$

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Show that $(a)\ \forall n \geq 1,\ \sum_{d/n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise} \end{cases}$

$(b)\ \lambda^{-1}(n) = |\mu(n)|$

$(c)\ \sigma_\alpha^{-1}(n) = \sum_{d/n} d^\alpha \mu(d)\mu(\frac{n}{d}).$

19. State and prove Euler's summation formula. Hence show that
$\forall\ x \geq 1,\ \sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$, if $s > 0, s \neq 1$ where ζ is the Remann zeta functon.

20. State and prove Shapiro's Tauberian Theorem.

21. State and prove quadratic reciprocity law for Legendre's symbol and hence determine whether 219 is a quadratic residue modulo 383.

(2 × 5 = 10 Weightage)

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