**23P302** (Pages: 2) Name: ...............................

Reg.No: ...............................

**THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2024**

(CBCSS - PG)

(Regular/Supplementary/Improvement)

**CC19P MTH3 C12 - COMPLEX ANALYSIS**

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

**Part A**

Answer ***all*** questions. Each question carries 1 weightage.

1. Find the fixed points of a translation and the inversion on $C_\infty$.
2. Show that a Mobius transformation takes circles onto circles.
3. Define an orientation for a circle $\Gamma$ of $C_\infty$. Also state the orientation principle.
4. Let $\gamma : [0, 2\pi] \to C$ given by $\gamma(t) = e^{it}$. Evaluate $\int_\gamma \frac{1}{z} dz$.
5. State first and second versions of Cauchy's integral formula.
6. When an isolated singularity is said to be removable? Prove that $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$.
7. Using residue theorem show that $\int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi}{2}$.
8. Suppose $f$ and $g$ are meromorphic in a neighbourhood of $\bar{B}(a; R)$ with no zeros or poles in the circle $\gamma = \{z : |z - a| = R\}$. If $Z_f, Z_g, P_f, P_g$ are the number of zeros and poles of $f$ and $g$ inside $\gamma$ counted according to their multiplicities and if $|f(z) + g(z)| \leq |f(z)| + |g(z)|$ on $\gamma$, prove that $Z_f - P_f = Z_g - P_g$.

**(8 × 1 = 8 Weightage)**

**Part B**

Answer any ***two*** questions each unit. Each question carries 2 weightage.

**UNIT - I**

9. Consider the stereographic projection between $C_\infty$ and $S = \{(x_1, x_2, x_3) \in R^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Let $z = x + iy \in C$ and $Z = (x_1, x_2, x_3)$ be the corresponding point of S. Express $Z = (x_1, x_2, x_3)$ in terms of $z$. Also prove that $d(z, z') = \frac{2|z - z'|}{\left[(1 + |z|^2)(1 + |z'|^2)\right]^{1/2}}$ for $z, z' \in C$.

10. If $\sum a_n(z-a)^n$ is a power series with radius of convergence $R$. Prove that $R = \lim \left|\frac{a_n}{a_{n+1}}\right|$. provided the limit exists.

11. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Prove that for each $k \geq 1$ the series $\sum_{n=k}^{\infty} n(n-1)(n-2)...(n-k+1)a_n(z-a)^{n-k}$ has radius of convergence $R$ and for $n \geq 0,\ a_n = \frac{1}{n!}f^{(n)}(a)$.

**UNIT - II**

12. Let $f: G \to C$ be analytic and suppose $\bar{B}(a;r) \subset G,$ where r is positive. Also let $\gamma(t) = a + re^{it},\ 0 \leq t \leq 2\pi$. Prove that $f(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}dw$ for $|z-a| < r$.

13. Let G be a connected open set and $f: G \to C$ be analytic. Prove that the following statements are equivalent,
(i) $f \equiv 0$ on G.
(ii) There is a point $a \in G$ such that $f^{(n)}(a) = 0$ for all non-negative integer $n$.
(iii) The set $\{z \in G: f(z) = 0\}$ has a limit point in G.

14. If G is simply connected and $f: G \to C$ is analytic in G. Prove that $f$ has a primitive in G.

**UNIT - III**

15. Find the residue at the poles of the function $f(z) = \frac{1}{z^4+1}$.

16. Prove that if G is a region and $f: G \to C$ is analytic function such that there is a point $a$ in G with $|f(a)| \geq |f(z)|$ for all $z$ in G,then $f$ is a constant.

17. Suppose $f$ is analytic on $D = \{z: |z| < 1\}$ and $|f(z)| \leq 1$ for $z \in D$. Prove that $|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}$ for $a \in D$. Also the equality holds exactly when $f(z) = \varphi_{-\alpha}(c\varphi_a(z)$ where $\alpha = f(a)$ for some $c \in C$ with $|c| = 1$ and $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$.

**(6 × 2 = 12 Weightage)**

**Part C**

Answer any ***two*** questions. Each question carries 5 weightage.

18. (a) Let G be either the whole plane C or some open disk and if $u: G \to R$ is a harmonic function. Show that $u$ has a harmonic conjugate.
(b) Show that the function $u(x, y) = e^x(x\cos y - y\sin y)$ is harmonic. Find its harmonic conjugate.

19. Let $\gamma : [a, b] \rightarrow C$ is of bounded variation and suppose that $f : [a, b] \rightarrow C$ is continuous. Prove that there is a complex number $\boldsymbol{I}$ such that for every $\epsilon > 0$ there is a $\delta > 0$ such that when $P = \left\{t_0 < t_1 < \ldots < t_m\right\}$ is a partition of $[a, b]$ with $\|P\| = max\left\{(t_k - t_{k-1}) : 1 \leq k \leq m\right\} < \delta$ then

$$\left| I - \sum_{k=1}^{m} f(\tau_k)\left[\gamma(t_k) - \gamma(t_{k-1})\right] \right| < \epsilon$$ for whatever choice of points $\tau_k,\ t_{k-1} \leq \tau_k \leq t_k$.

20. Let G be an open set and $f : G \rightarrow C$ be a differentiable function. Prove that $f$ is analytic on G.

21. Let $f(z) = \dfrac{1}{z(z-1)(z-2)}$. Give the Laurent series expansion of *f(z)* in each of the annuli:

(a) $ann(0, 0, 1)$; (b) $ann(0, 1, 2)$; (c) $ann(0, 2, \infty)$.

**(2 × 5 = 10 Weightage)**

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