23P303 (Pages: 2) Name: ....

Reg.No: ....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2024

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH3 C13 - FUNCTIONAL ANALYSIS

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Define subspace of a linear space. Show that if $A : E_1 \to E_2$ is a linear map then $\text{Im } A$ is a subspace of E_2.
2. Define Cauchy sequence in a normed space X. Prove that if x_n is a Cauchy sequence, then prove that it is a bounded sequence.
3. Prove that inner product is a continuous function.
4. Show that if f_i is a complete system in a Hilbert space H and $x \perp f_i$, then $x = 0$.
5. Show that for every closed subspace of H,$L \oplus L^{\perp} = H$.
6. Define a bounded functional. If f is a bounded linear functional then prove that $|f(x)| \leq \|f\|^{*}\|x\|$.
7. Show that any bounded set in $\mathbb{R}^n$ is relatively compact.
8. Define Norm convergence and strong convergence in $L(X)$.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. Define a norm on $c[0, 1]$ and prove that it is a norm.
10. Let O be an openset then prove that $F = O^c$ closed . Also prove if F is closed set then F^c is open.
11. Let X_0 be a closed subspace of X.. Verify X/X_0 is a normed space together with the norm defined by $||[x]|| = \inf_{y \in X_0} ||x - y||$

UNIT - II

12. Show that for any $x \in H$ and any orthonormal system $\{e_i\}_1^{\infty}$, there exists a $y \in H$ such that $y = \sum_{i=1}^{\infty} < x, e_i > e_i$
13. Let M be a convex closed set in H. Show that there exists a unique $y \in M$ such that $\rho(x, M) = \|x - y\|$

14. Consider $f \in E^{\#} - \{0\}$. Then show that

   1. Codim ker $f = 1$
   2. If $f, g \in E^{\#} - \{0\}$ and ker $f =$ ker g, then there exists $\lambda \neq 0$ such that $\lambda f = g$.

UNIT - III

15. Prove that for any normed space X, the dual space X^* is always complete.

16. Let $L \hookrightarrow X$ be a subspace of a normed space X and let $x \in X$ such that dist $(x, L) = d > 0$. Then, prove that there exists $f \in X^*$ such that $\|f\| = 1$, $f(L) = 0$ and $f(x) = d$.

17. If $\|A\| = q < 1$, then prove that $(I - A)$ is invertible and $(I - A)^{-1} = \sum_0^\infty A^k$. More over $\|(I - A)^{-1}\| \leq \dfrac{1}{1 - \|A\|}$.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Prove that the Hilbertspace is seperable if and only if there exist a complete orthonormal system $\{e_i\}_{i \geq 1}$

19. Let X be a normed space and let Y be a complete normed space. Then show that $L(X \to Y)$ is a Banach Space.

20. Prove that M is relatively compact if and only if for every $\varepsilon > 0$, there exists a finite ε- net in M.

21. State and prove $H\ddot{o}lder's$ inequality and Minkowski's inequality for the scalar sequences.

(2 × 5 = 10 Weightage)

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