23P403 (Pages: 2) Name: ....

Reg.No: ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2025

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 E05 - ADVANCED COMPLEX ANALYSIS

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Show that the set of all analytic functions on a region G, $H(G)$ is a complete metric space under the metric ρ.
2. Prove that $H(G) \cup \{\infty\}$ is closed in $C(G, \mathbb{C}_\infty)$.
3. Show that Conformal equivalence is an equivalence is an equivalence.
4. Let $\{z_n\}$ be a sequence of nonzero complex numbers. Suppose $\prod_{k=1}^{\infty} z_k$ exists with $\prod_{k=1}^{\infty} z_k \neq 0$. Prove that $\lim_{n\to\infty} z_n = 1$.
5. Define the elementary factor function $E_p(z)$. Show that $E_p(z/a)$ has a simple zero at $z = a$ and no other zero.
6. Show that $\log \Gamma(x)$ is a convex function for $x > 0$.
7. Prove that the constant term of the Laurent series of $(e^z - 1)^{-1}$ about $z = 0$ is $-\frac{1}{2}$
8. When an entire function is said to be of finite order? Also define the order of an entire function *f*.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. Suppose G is open in $\mathbb{C}$. Prove that there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \cup_{n=1}^{\infty} K_n$ satisfying
   $K_n \subseteq \text{int } K_{n+1}$
   Every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$.
10. Show that $\mathcal{F} \in H(G)$ is normal if and only if $\mathcal{F}$ is locally bounded.
11. Let (X_n, d_n) are metric spaces for each *n*. Prove that the space $\left(\prod_{n=1}^{\infty} X_n, d\right)$ where $d = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^n \left(\frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}\right)\right]$ is a metric space.

UNIT - II

12. Show that $\cos \pi z = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right)$. Using factorization of $\cos \pi z$, find a factorization for $\cosh \pi z$

13. Let K be a compact subset of $\mathbb{C}$ and let E be a subset of $\mathbb{C}_\infty - K$ that meets each component of $\mathbb{C}_\infty - K$.If f is analytic on an open set containing K and $\varepsilon > 0$. Prove that there is a rational function $R(z)$ whose only poles lie in E and $|f(z) - R(z)| < \varepsilon$ for all z in K.

14. Show that $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ for z not an integer. Deduce $\Gamma(1/2) = \sqrt{\pi}$

UNIT - III

15. State and prove Schwarz reflection principle.

16. Let (f, D) be a function element and let G be a region containing D such that (f, D) admits unrestricted continuation in G. Let $a \in D,\ b \in G$ and γ_0 and γ_1 be paths in G from a to b. Also let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ and $\{(g_t, D_t) : 0 \leq t \leq 1\}$ be analytic continuations of (f, D) along $\gamma_0$and γ_1 respectively. If γ_0 and γ_1 are FEP homotopic, prove that $[f_1]_b = [g_1]_b$

17. Let f be an entire function, $M(r) = sup\{|f(re^{i\theta}| : 0 \leq \theta \leq 2\pi\}$ and $n(r)$ is the number of zeros of f in $B(0;r)$ counted according to multiplicity. Suppose $f(0) = 1$. Prove that $n(r)\log 2 \leq \log M(2r)$.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Show that a set $\mathcal{F} \subseteq (C(G, \Omega))$ is normal iff the following two conditions are satisfied.
for each $z \in G,\ \{f(z) : f \in \mathcal{F}\}$ has compact closure in Ω.
$\mathcal{F}$ is equicontinuous at each point of G

19. Prove that 1. $\left(1 - \frac{t}{n}\right)^n \leq e^{-t}$ for $t \geq 0$ and $n \geq t$.

2. $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ for $Re(z) > 0$.

20. (a) If $Re(z) > 1$, then prove that $\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty}\left(\int_0^\infty e^{-nt}t^{z-1}dt\right)$

(b) Show that $\zeta(z)\Gamma(z) = \int_0^\infty (e^t - 1)^{-1}t^{z-1}dt$ for $Re(z) > 1$.

21. Let f be an entire function of finite order λ and $\varepsilon > 0$. Prove that $|f(z)| < exp(|z|^{\lambda+\varepsilon})$ for all z with $|z|$ sufficiently large and an z can be found with $|z|$ as large as desired so that $|f(z)| \geq exp(|z|^{\lambda-\varepsilon})$

(2 × 5 = 10 Weightage)

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