23P401 (Pages: 2) Name: ....

Reg.No: ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2025

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 E08 - COMMUTATIVE ALGEBRA

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Define ring homomorphism and kernel and image of a hmomorphism.
2. Define extension and contraction of an ideal.
3. Define modules with an example.
4. Let M be an A-module. If $M=0$, then prove that $M_p = S^{-1}M = 0$, for all prime ideal p of A (Here $S = A - p$).
5. Let q be a p-primary ideal of Ring A and $x \in A$. If $x \in q$, then prove that $(q : x) = \langle 1 \rangle$.
6. Define Noetherian module and Artinian module.
7. Let A be Noetherian, a an ideal of A. Then prove that A/a is Noetherian ring.
8. Prove that in an Artin ring nilradical is equal to the Jacobson radical.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. If $L \supseteq M \supseteq N$ are A-modules. Then prove that $(L/N)/(M/N) \cong L/M$.
10. State and prove Nakayama's lemma.
11. Let M, N and P are A-modules. Then prove that there exist a unique isomorphism from $(M \otimes N) \otimes P$ to $M \otimes N \otimes P$.

UNIT - II

12. Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B, for all $s \in S$. Then prove that there exists a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$, where $f : A \to S^{-1}A$ defined by $f(x) = x/1$.
13. Prove that the $S^{-1}A$-modules $S^{-1}(M/N)$ and $S^{-1}M/S^{-1}N$ are isomorphic.

14. Prove that $S^{-1}(r(a)) = r\left(S^{-1}(a)\right)$, for any ideal a in A.

UNIT - III

15. If $A[x]$ is contained in a sub ring C of B such that C is finitely generated A-module, then prove that there exists a faithful $A[x]$-module M which is finitely generated as an A-module.

16. Prove that if A is Noetherian, so is $A[x_1, x_2, \ldots, x_n]$.

17. Prove that in a Noetherian ring, every irreducible ideal is primary.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. (i) Prove that every ring $A \neq 0$ has atleast one maximal ideal.

    (ii) Let A be a ring and m, a maximal ideal of A such that every element of $1 + m$ is a unit in A. Then prove that A is a local ring.

19. (i) Prove that the nilradical of A is the intersection of all prime ideals of A.

    (ii) Prove that x belongs to Jacobson Radical of A if and only if $1 - xy$ is a unit in A for all $y \in A$.

20. (i) Let $a = \bigcap_{i=1}^{n} q_i$ be a minimal primary decomposition of a decomposible ideal a. Let $p_i = r(q_i), 1 \leq\leq i \leq n$. Then prove that the p_i are precisely the prime ideals which occur in the set of ideals $r(a : x), x \in A$ and hence independent of the particular decomposition of a.

    (ii) If the zero ideal of A is decomposible, then prove that the set D of zero divisors of A is the union of the prime ideals belonging to zero ideal.

21. (i) Let $A \subseteq B$ be integral domains, B is integral over A. Then prove that B is a field if and only if A is a field.

    (ii) State and prove going-up theorem.

(2 × 5 = 10 Weightage)

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