23P404 (Pages: 2) Name: ....

Reg.No: ....

FOURTH SEMESTER M.Sc. DEGREE EXAMINATION, APRIL 2025

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19P MTH4 E09 - DIFFERENTIAL GEOMETRY

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Show that the graph of any function $f : R^n \to R$ is a level set for some function $F : R^{n+1} \to R$.
2. Let S be an $n-1$ - surface in $\mathbb{R}^n$ given by $S = f^{-1}(c)$ where $f : U \to R, (U$ open in $\mathbb{R}^n)$ is such that $\nabla f(p) \neq 0$ for all $p \in f^{-1}(c)$. Let $g : U_1 \to R$, defined by $g(x_1, x_2, \dots . x_{n+1}) = f(x_1, x_2, .., x_n)$ where $U_1 = U \times R$. Then prove that $g^{-1}(c)$ is an n-surface in $\mathbb{R}^{n+1}$.
3. Write the statement of Lagrange multiplier theorem.
4. Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed, then $\alpha(\dot{t}) \perp \alpha(\ddot{t})$ for every $t \in I$.
5. Let S be an n- surface in R^{n+1} and $\alpha : I \to S$ be a parametrised curve on S. Then for smooth vector fields $\mathbf{X}$ and $\mathbf{Y}$, prove that $(\mathbf{X}+\mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$
6. Compute $\nabla_v f$, where $f : R^{n+1} \to R$, $v \in R_p^{n+1}$, where $f(x_1, x_2) = 2{x_1}^2 + 3{x_2}^2$, $v = (1, 0,\ 2, 1), n = 1$.
7. If β is any reparametrisation of α, then $\int_\alpha \omega = \int_\beta \omega$.
8. Define a surface of dimension n in $R^{n+k}, (k \geq 1)$.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions each unit. Each question carries 2 weightage.

UNIT - I

9. Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f, and let $c = f(p)$. Then the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^\perp$.
10. Let $S \subset \mathbb{R}^{n+1}$ be a connected n- surface in $\mathbb{R}^{n+1}$. Then prove that there exists exactly two orientations on S.
11. Find the integral curve of the vector field $\mathbf{X}(p) = (p, X(p))$ where $X(x_1, x_2) = (-x_2, x_1)$ passing through $(i)(1, 0)$ and $(ii)(a, b)$.

UNIT - II

12. Let $\mathbf{X}$ and $\mathbf{Y}$ be any two smooth vector fields on U and $f : U \to R$ be a smooth function and let $p \in U,\ v \in R_p^{n+1}$, then prove that (i) $\nabla_v(\mathbf{X}+\mathbf{Y}) = \nabla_v(\mathbf{X}) + \nabla_v(\mathbf{Y})$ and (ii) $\nabla_v(f\,\mathbf{X}) = \nabla_v(f)\mathbf{X}(p) + f(p)\nabla_v(\mathbf{X})$.

13. Find the global parametrization of the curve $(x_1 - a)^2 + (x_2 - b)^2 = r^2$.

14. Find the length of the connected oriented plane curve $f^{-1}(c)$ oriented by $\frac{\nabla f}{\|\nabla f\|}$ where $f : U \to R$ is given by $f(x_1, x_2) = 5x_1 + 12x_2,\ U = \{(x_1, x_2) : {x_1}^2 + {x_2}^2 < 169\}, c = 0$.

UNIT - III

15. Let V be a finite dimensional vector space with dot product and let $L : V \to V$ be a self adjoint linear transformation on V. Then prove that there exists an orthonormal basis for V consisting of eigen vectors of L.

16. On each compact oriented n- surface S in $\mathbb{R}^{n+1}$, prove that there exists a point p such that the second fundamental form at p is definite.

17. If $L : R^n \to R^{n+1} (k \geq 1)$ is a non singular linear map and let $w \in R^{n+k}$, then show that the map $\psi : R^n \to R^{n+k}$ defined by $\psi(p) = L(p) + w$ is a parametrised n- surface (n- plane) through w in R^{n+k}.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Let X be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$, and let $p \in U$. Then, prove that there exists an open interval I containing 0 and an integral curve $\alpha : I \to U$ of X such that: (i) $\alpha(0) = p$, (ii) If $\beta : \tilde{I} \to U$ is any other integral curve of X with $\beta(0) = p$, then $\tilde{I} \subset I$, and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

19. Let S be an n- surface in $\mathbb{R}^{n+1}$, let X be a smooth tangent vector field on S, and let $p \in S$. Then prove that there exists an open interval I containing 0 and a parametrized curve $\alpha : I \to S$ such that:
(i) $\alpha(0) = p$, (ii) $\dot{\alpha}(t) = X(\alpha(t))$, for every $t \in I$ (iii) If $\beta : \tilde{I} \to S$ is any other parametrized curve in S satisfying (i) and (ii), then prove that $\tilde{I} \subset I$, and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.

20. Let S be an S surface in $\mathbb{R}^{n+1}$, let $p \in S$ and $v \in S_p$. Then prove the existence and uniqueness of the maximal geodesic in S passing through p with initial velocity v.

21. State and prove inverse function theorem for n- surfaces. Hence deduce that if S is a compact connected oriented n- surface in R^{n+1} whose Gauss Kronecker curvature is nowhere zero, then the Gauss map $N : S \to S^n$ is a diffeomorphism.

(2 × 5 = 10 Weightage)

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