25P105 (Pages: 2) Name: ....

Reg.No: ....

FIRST SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2025

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19PMTH1C05 - NUMBER THEORY

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Prove that $\forall\ n \geq 1, \sum\limits_{d/n} \Lambda(d) = \log n$.
2. If the arithmetical functions f and g are completely multiplicative, then check whether fg and f/g, where $g(n) \neq 0, \forall n$ are completely multiplicative.
3. Prove that $\forall\ x \geq 1,\ \sum\limits_{n \leq x} \frac{1}{n} = \log x + C + O(1/x)$, where C is the Euler's constant.
4. Prove that for $x \geq 2, \pi(x) = \frac{\tau(x)}{\log x} + \int\limits_2^x \frac{\tau(t)}{t \log^2 t} dt$.
5. Prove that $\forall\ x \geq 1,\ \sum\limits_{n \leq x} \psi(\frac{x}{n}) = x \log x - x + O(\log x)$.
6. Determine whether 888 is a quadratic residue or non-residue mod 1999.
7. Find the inverse of the matrix $\begin{bmatrix} 40 & 0 \\ 0 & 21 \end{bmatrix} (\text{mod } 841)$.
8. How to send a digital signature in RSA cryptosystem?

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT - I

9. (a) Prove that $\forall n \geq 1, \sum\limits_{d/n} \phi(d) = n$.

   (b) Find all integers n such that $\phi(n) = \frac{n}{2}$.

10. Prove that $\Lambda(n) \log n + \sum\limits_{d/n} \Lambda(d)\Lambda(\frac{n}{d}) = \sum\limits_{d/n} \mu(d) \log^2(\frac{n}{d})$.

11. If a and b are positive real numbers such that $a.b = x$, then show that $\sum_{\substack{q,d \\ qd \leq x}} f(d)g(q) = \sum_{n \leq a} f(n)G(\frac{x}{n}) + \sum_{n \leq a} g(n)F(\frac{x}{n}) - F(a)G(b)$.

UNIT - II

12. Show that for $x > 0,\ 0 \leq \frac{\psi(x)}{x} - \frac{\tau(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x}\log 2}$

13. Let $\{a(n)\}$ be a non-negative sequence such that $\sum_{n \leq x} a(n)[\frac{x}{n}] = x\log x + O(x),\ \forall\, x \geq 1$. Then prove that there exist a constant $A > 0$ and an $x_0 > 0$ such that $\sum_{n \leq x} a(n) \geq Ax,\ \forall\, x \geq x_0$.

14. Show that there is a constant A such that $\sum_{p \leq x} \frac{1}{p} = \log\log x + A + O(\frac{1}{\log x}),\ \forall\, x \geq 2$.

UNIT - III

15. For every odd prime p, prove that $(-1|p) = (-1)^{\frac{p-1}{2}}$ and $(2|p) = (-1)^{\frac{p^2-1}{8}}$.

16. Define shift cryptosystem and affine cryptosystem. Also find the plain text corresponding to the ciphertext ZXNGA in the affine cryptosystem with the enciphering key (7, 3) in the 26- letter alphabet system.

17. Solve the system: $x + 3y \equiv 1(\mathrm{mod}\ \ 26)$
$7x + 9y \equiv 1(\mathrm{mod}\ \ 26)$

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. Prove that the set G of all arithmetical functions f with $f(1) \neq 0$ is an abelian group with respect to Dirichlet multiplication.

19. Show that $(a)\ \forall n \geq 1,\ \sum_{d/n} \lambda(d) = \begin{cases} 1, & \text{if } n \text{ is a square} \\ 0, & \text{otherwise} \end{cases}$ $\quad (b)\ \lambda^{-1}(n) = |\mu(n)|$
$(c)\ \sigma_\alpha^{-1}(n) = \sum_{d/n} d^\alpha \mu(d)\mu(\frac{n}{d})$.

20. Prove that the following relations are logically equivalent:
$(a)\ \lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$ $\quad (b)\ \lim_{x\to\infty} \frac{\pi(x)\log \pi(x)}{x} = 1$ $\quad (c)\ \lim_{n\to\infty} \frac{P_n}{n\log n} = 1$

21. State and prove Euler's criterion for Legendre's symbol. Also check whether 6 is a quadratic residue modulo 23.

(2 × 5 = 10 Weightage)

****