24P302 (Pages: 2) Name: ....

Reg.No: ....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, NOVEMBER 2025

(CBCSS - PG)

(Regular/Supplementary/Improvement)

CC19PMTH3C12 - COMPLEX ANALYSIS

(Mathematics)

(2019 Admission onwards)

Time : 3 Hours Maximum : 30 Weightage

Part A

Answer ***all*** questions. Each question carries 1 weightage.

1. Find the fixed points of a dilation, a translation and the inversion on $\mathbb{C}_\infty$.
2. State and prove symmetry principle.
3. Let G be open in $\mathbb{C}$ and let γ be a closed rectifiable path in G. Also if $f : G \to \mathbb{C}$ is continuous with a primitive $F : G \to \mathbb{C}$. Prove that $\int_\gamma f = 0$.
4. State and prove Cauchy's estimate.
5. Suppose $f : G \to \mathbb{C}$ is one-one, analytic and $f(G) = \Omega$. Prove that $f^{-1} : \Omega \to \mathbb{C}$ is analytic and $\left(f^{-1}\right)'(w) = [f'(z)]^{-1}$ where $w = f(z)$.
6. Determine the poles of the function $f(z) = \dfrac{1}{z^4+1}$.
7. Suppose f has a pole of order m at $z = a$ and put $g(z) = (z-a)^m f(z)$. Prove that $Res(f;a) = \dfrac{1}{(m-1)!} g^{(m-1)}(a)$.
8. Prove that if *f* is analytic in a region G and '*a*' is a point in G with $|f(a)| \geq |f(z)|,\ \forall z \in G$, then *f* is a constant.

(8 × 1 = 8 Weightage)

Part B

Answer any ***two*** questions from each unit. Each question carries 2 weightage.

UNIT - I

9. Consider the stereographic projection between $\mathbb{C}_\infty$ and $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Let $z = x + iy \in \mathbb{C}$ and $Z = (x_1, x_2, x_3)$ be the corresponding point of S. Express $Z = (x_1, x_2, x_3)$ in terms of *z*. Also prove that $d(z, z') = \dfrac{2|z - z'|}{\left[(1+|z|^2)(1+|z'|^2)\right]^{1/2}}$ for $z, z' \in \mathbb{C}$.

10. For a given power series $\sum_{n=0}^{\infty} a_n(z-a)^n$, let $\frac{1}{R} = \limsup |a_n|^{1/n},\ 0 \le R \le \infty$. Prove the following

1. If $|z-a| < R$, the series converges absolutely.
2. If $0 < r < R$, the series converges uniformly on $\{z : |z| \le r\}$.

11. Let f and g be analytic on G and Ω respectively and suppose that $f(G) \subseteq \Omega$. Prove that $g \circ f$ is analytic on G and $(g \circ f)'(z) = g'(f(z))f'(z)$.

UNIT - II

12. Evaluate $\int_\gamma \frac{z^2+1}{z(z^2+4)}dz$ where $\gamma(t) = re^{it},\ 0 \le t \le 2\pi$ for all possible values of r, $0 < r < 2,\ and\ 2 < r < \infty.$

13. Let G be a connected open set and $f : G \to \mathbb{C}$ be analytic. Prove that the following statements are equivalent,

1. $f \equiv 0$ on G.
2. There is a point $a \in G$ such that $f^{(n)}(a) = 0$ for all non-negative integer n.
3. The set $\{z \in G : f(z) = 0\}$ has a limit point in G.

14. State and prove the first version of Cauchy's integral formula.

UNIT - III

15. State and prove argument principle.

16. State and prove Schwarz's lemma.

17. If f has an essential singularity at $z = a$, prove that for every $\delta > 0,\ \{f[ann(a; 0, \delta)]\}^{-} = \mathbb{C}$.

(6 × 2 = 12 Weightage)

Part C

Answer any ***two*** questions. Each question carries 5 weightage.

18. (a) Let G be either the whole plane $\mathbb{C}$ or some open disk and if $u : G \to \mathbb{R}$ is a harmonic function. Show that u has a harmonic conjugate.

(b) Show that the function $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire plane. Find its harmonic conjugate.

19. Prove that the cross ratio of four distinct points is real if and only if all the four points lie on a circle. Also if S be a Mobius transformation and Γ be any circle of $\mathbb{C}_\infty$, show that $S(\Gamma)$ is also a circle of $\mathbb{C}_\infty$.

20. State and prove third version of Cauchy's theorem.

21. Evaluate $\int_0^\infty \frac{\sin x}{x}dx.$

(2 × 5 = 10 Weightage)

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